Two-loop world-sheet corrections in $AdS_5 \times S^5$ superstring

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ABSTRACT: We initiate the computation of the 2-loop quantum $AdS_5 \times S^5$ string corrections on the example of a certain string configuration in S^5 related by an analytic continuation to a folded rotating string in AdS_5 in the "long string" limit. The 2-loop term in the energy of the latter should represent the subleading strong-coupling correction to the cusp anomalous dimension and thus provide a further check of recent conjectures about the exact structure of the Bethe ansatz underlying the AdS/CFT duality. We use the conformal gauge and several choices of the κ-symmetry gauge. We present partial results: we compute the bosonic contribution to the effective action and also determine the transcendental form of the fermionic contribution but do not succeed in verifying the cancellation of all logarithmic divergences. The main obstacle appears to be an apparent non-renormalizability of the GS superstring (when expanded near a non-trivial background) action which first manifests itself at the 2-loop order and leads to subtleties in dealing with power divergent integrals.

KEYWORDS: AdS-CFT Correspondence, Field Theories in Lower Dimensions.

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Contents

1. Introduction

To demonstrate the AdS/CFT duality one is to establish a direct equivalence between the spectrum of the $\mathcal{N} = 4$ SYM dilatation operator and the spectrum of quantum string energies in $AdS_5 \times S^5$. There are strong indications that both spectra are indeed described by solutions of certain Bethe ansätze (for a recent review and some references see, e.g., [1]).

While the gauge-theory side of the duality has standard definition at weak-coupling, the presence of the RR background supporting $AdS_5 \times S^5$ requires that the formulation of the dual string theory should be based on the manifestly-supersymmetric Green-Schwarz approach [2, 3] which leads to a complicated-looking non-linear action [4 – 6].

The quantization of this action is straightforward at leading semiclassical (1-loop) order by expanding near a non-trivial classical string configuration and fixing an appropriate κ-symmetry gauge (see, e.g., [7 – 10]). This allowed one to compute 1-loop string corrections to energies of various classical solutions in $AdS_5 \times S^5$ [10 – 15], and these explicit results played a key role in checking the AdS/CFT duality and, in particular, in recent progress in fixing the structure of the "string" (strong-coupling) form [16] of the Bethe ansatz [17 – 20] which led to the exact expressions in [21, 1].[1]

To provide further important checks of the conjectured form of the Bethe ansatz for the gauge/string spectrum it is crucial to learn how to extend the 1-loop computations of [10]–[15] beyond the 1-loop level. Here, however, one faces an apparent problem: the curved-space GS action expanded near a string background that provides the fermions with a non-trivial propagator is formally non-renormalizable beyond one loop. While the original string action has no dimensional parameters and both the bosonic and the fermionic fields in it are dimensionless, when expanding near a non-trivial background one effectively changes the dimension of fermions to canonical Dirac field one (1/2) in 2 dimensions. The effective dimensional scale is introduced by the derivative of the bosonic string background, leading to non-renormalizable couplings (and thus to higher power divergences).[2]

This problem did not seem to be appreciated in early studies of quantum GS action which were restricted to 1-loop order [24], but it was recently emphasized in [25], where it was suggested that it may be possible to resolve it in a special "light-cone"-type gauge. On general grounds, one should not expect any meaningful results to depend on a particular gauge choice, but the formulation of quantum theory may look simpler in a gauge where the action has less non-linear form (e.g. being quadratic in l.c. gauge in flat space).[3]

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[1]An additional input was the assumption of crossing symmetry [22, 23].

[2]It is sometimes said that one cannot quantize GS action since fermions θ "do not have a propagator". This is somewhat a misleading statement. The quantization of the $AdS_5 \times S^5$ action is formally well-defined as soon as one chooses a non-trivial bosonic background near which one can expand the action (and fixes a proper κ-symmetry gauge). There is an analogy with the quantization of Einstein's theory: unless one chooses a non-zero background metric the metric fluctuations do not have a propagator term — the Einstein action is non-polynomial in the metric. Specifying a background metric introduces a dimensional coupling and also spontaneously breaks the diffeomorphism invariance of the Einstein action; it can be formally maintained using the background field method in which the background metric is also transforming (provided one uses a background-covariant gauge). Similar approach can be followed for the GS string. In most practical applications (see, e.g., [10, 12]) one needs to expand near a specific background which spontaneously breaks symmetries of the original action, just as in a generic case of the semiclassical expansion near a solitonic solution.

[3]A possible alternative is to use the Berkovits version of the $AdS_5 \times S^5$ GS action [26] that has a non-degenerate fermionic quadratic term from the start and formally defines a renormalizable theory. However, the formulation of the theory (using BRST symmetry as a basic principle) is somewhat *ad hoc* and is not completely free of ambiguities (in particular, in the definition of the ghost path integral measure). To see if this formulation is of practical use for addressing the issues discussed here it would be important to first

On general grounds, one should expect the GS action to make sense at the quantum level only if it happens to be UV finite: this is required by its basic gauge symmetry — the κ-symmetry. The key technical issue is how to formulate the quantum theory (i.e. make a choice of a regularization, measure, etc.) in a way that is indeed consistent with the preservation of the classical symmetries at the quantum level.[4]

Our aim here will be to begin the investigation of the quantum $AdS_5 \times S^5$ string theory beyond the 1-loop order by attempting to compute a 2-loop correction to the string world-sheet effective action in a particular string background. This background appears to be one of the simplest possible non-trivial choices, making the 2-loop computation tractable. It may be viewed as a particular limit of the circular string solution with two equal SO(6) spins [11, 27] and is an example of a "homogeneous" spinning string solution for which the only non-vanishing string coordinates are isometric angles of $AdS_5 \times S^5$ which are linear in string world-sheet coordinates τ and σ. This choice is special in that, when expanded near it, the $AdS_5 \times S^5$ string Lagrangian has *constant* (τ, σ independent) coefficients and thus the computation of quantum corrections simplifies considerably. An apparent problem, however, is that the simplest spinning string solution with two equal SO(6) spins [11, 27] is unstable, and that seems to lead to potential problems in trying to compute the 2-loop correction to its energy.[5] One may avoid this instability by a formal analytic continuation in the winding number m, i.e. by taking it less than one or even purely imaginary.

Remarkably, there is also another important reason to study quantum corrections to the energy of the circular 2-spin S^5 solution with an imaginary winding parameter. As was noticed recently [15] (using an earlier observation in [28]), this solution is related by a formal analytic continuation to a "long-string" limit of the folded string rotating in AdS_5 with spin S and also orbiting along big circle of S^5 with spin J. The energy of the $S \gg J$ string [30, 10] goes as $S + a_0 \sqrt{\lambda} \ln \frac{S}{J}$ in the "long-string" limit, and it played a key role in recent discussions of the AdS/CFT correspondence in the SL(2) sector [1, 31−35].

Let us start with introducing the relevant string background and reviewing the form of the 1-loop correction to its energy.

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reproduce the results of the 1-loop GS computations in [10]–[15] by starting with the Berkovits action.

[4]By this we mean, in particular, that the κ-symmetry does not develop anomalies, i.e. anomalies cancel. The usual quantization schemes specify a regulator that preserves as many symmetries as possible. Anomalies may arise, however, if a symmetry is broken by the regulator. A formal argument for finiteness of the $AdS_5 \times S^5$ action [4] constructed by analogy with the one for the WZW theory runs as follows: (i) the "kinetic" term in the action is protected by global symmetry (as for, e.g., SO(n) coset sigma model) and can thus be renormalised only by an overall factor; (ii) the coefficient of the WZ term in the action of [4] cannot be renormalised (for a symmetric supercoset the analog of the field strength of the B_{mn} coupling is covariantly constant; alternatively, the WZ term has a 3d representation that is not possible for local covariant counterterms); (iii) the κ-symmetry relates the coefficients of the WZ and the "kinetic" terms, thus precluding any renormalization of the latter. This argument is very formal since it assumes that both global supercoset symmetry and the κ-symmetry are actually preserved at the quantum level. The main issue is how to formulate the quantum theory explicitly so that these conditions are indeed met.

[5]A similar "homogeneous" circular string solution with one spin in AdS_5 and one in S^5 *is* stable, but the corresponding fluctuation spectrum (and thus the propagator) is much more involved [14], substantially complicating the problem of computing the 2-loop correction.

1.1 String background and strong-coupling expansion of minimal twist anomalous dimension

According to [30, 10] the classical energy of a folded rotating string in $AdS_5 \times S^5$ which should be dual to a minimal twist operator in planar $\mathcal{N} = 4$ SYM theory scales for $S \gg J$ as[6]

$$E = S + f(\lambda) \ln S + \cdots . \qquad (1.1)$$

For small λ the function $f(\lambda)$ should have the standard perturbative gauge theory expansion $f(\lambda) = k_1 \lambda + k_2 \lambda^2 + \cdots$ while for large λ it should have perturbative string theory expansion

$$f(\lambda) = a_0 \sqrt{\lambda} + a_1 + \frac{a_2}{\sqrt{\lambda}} + \cdots , \qquad a_0 = \frac{1}{\pi} , \quad a_1 = -\frac{3 \ln 2}{\pi} . \qquad (1.2)$$

The leading strong-coupling coefficients a_0 [30] and a_1 [10] were found to be in perfect agreement [1, 31, 32] with the prediction of the integral equation for the minimal twist anomalous dimension as extracted from the weak-coupling Bethe ansatz suggested in [1].[7]

It is obviously important to compute the value of the subleading coefficient a_2 directly as the *two-loop* correction in the $AdS_5 \times S^5$ string theory. It can then be compared with a prediction of [32] obtained numerically from the strong-coupling expansion of the solution of the integral equation of [1]:[8]

$$a_2 \approx -0.29154 \pm 0.0013 . \qquad (1.3)$$

In general, computing quantum corrections to the energy of the folded string solution in AdS [36, 30] is very complicated due to the non-trivial σ-dependent form of this configuration. However, as was realized in [10, 15] to extract the leading large spin $\frac{S}{\sqrt{\lambda}} \gg 1$ behaviour of the energy it is sufficient to consider the "long string" approximation in which the folded string solution simplifies, becoming effectively "homogeneous". Viewed as a string configuration in $AdS_3 \times S^1$ (where S^1 is from S^5) with the metric

$$ds^2 = d\rho^2 - \cosh^2 \rho \ dt^2 + \sinh^2 \rho \ d\theta^2 + d\phi^2 \qquad (1.4)$$

it is then approximated (in conformal gauge) by

$$t = \kappa\tau , \qquad \theta \approx \kappa\tau , \qquad \rho \approx \ell\sigma , \qquad \phi = \nu\tau , \qquad \ell \equiv \sqrt{\kappa^2 - \nu^2} , \qquad (1.5)$$

where S is related to κ and $J = \sqrt{\lambda}\nu$. The relevant limit we are interested in is

$$\kappa \gg 1 , \qquad \frac{\nu}{\kappa} = \text{fixed} \ll 1 \qquad (1.6)$$

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[6]More precisely, one is to assume that $\ln \frac{S}{J} \gg \frac{J}{\sqrt{\lambda}}$; we also omit a term linear in J on the r.h.s.

[7]This may not be totally surprising since the 1-loop dressing phase in the strong coupling Bethe ansatz was extracted [19] from other 1-loop string results; nevertheless, it provides a non-trivial check of the analytic continuation prescription suggested in [1], as it implies the existence of a single function with correct weak-coupling and strong-coupling limits.

[8]The strong-coupling expansion of this integrals equation appears to be subtle [34], so it is not clear to us if the numerical result of [32] for a_2 is completely without doubt; it would surely be important to obtain the expressions for the strong-coupling coefficients $a_1, a_2, \ldots$ analytically. Note that the coupling g used in [1, 32] is related to λ used here by $g = \frac{\sqrt{\lambda}}{4\pi}$.

which is sufficient for computing the coefficient of the leading $\ln S$ term in the energy.

The above configuration (1.5) is related [15] by a formal analytic continuation [28] to the $J_1 = J_2$ circular string solution in $R_t \times S^3$ part of $AdS_5 \times S^5$

$$ds^2 = -dt'^2 + d\psi^2 + \cos^2 \psi \, d\phi_2^2 + \sin^2 \psi \, d\phi_3^2 \tag{1.7}$$

taken in its form given in [11] ($J_1 = J_2 = \sqrt{\lambda} w$):

$$t' = \kappa'\tau, \qquad \psi = m\sigma, \qquad \phi_2 = \phi_3 = w\tau, \qquad w = \sqrt{\kappa'^2 - m^2} \,. \tag{1.8}$$

Under the continuation $t \to \phi_2$, $\rho \to i\psi$, $\phi \to \phi_3$, $\phi \to t'$ (one is also to change the overall sign of the string action) and the parameters are related as follows:

$$\kappa' = \nu \,, \qquad m = i\ell = i\sqrt{\kappa^2 - \nu^2} \,, \qquad w = \kappa \,. \tag{1.9}$$

The quadratic fluctuation action near the above solution will have constant coefficients after a coordinate rotation [11, 12]. We may also start directly with the same background (1.8) in the equivalent "rotated" form given in [27]:

$$\psi = \frac{\pi}{4} \,, \qquad \phi_2 = w\tau + m\sigma \,, \qquad \phi_3 = w\tau - m\sigma \,. \tag{1.10}$$

Then all coefficients in the fluctuation Lagrangian will be manifestly constant. It is the configuration (1.10) that will be our starting point for the quantum loop computation.

Our aim below will be to compute the 2-loop string correction to the energy of the circular solution (1.10) assuming the analytic continuation in m (1.9) and the scaling limit (1.6). For simplicity we shall also set $\nu = 0$, i.e. set the S^5 spin of the folded string to be zero or $\kappa' = 0$ for the rotating solution in (1.9):

$$\kappa' = \nu = 0 \,, \qquad m = -i\kappa \,, \qquad w = \kappa \,, \qquad \kappa \approx \frac{1}{\pi} \ln \frac{S}{\sqrt{\lambda}} \to \infty \,. \tag{1.11}$$

In this case the world sheet coordinates τ and σ in (1.10) can be rescaled by κ and since $\kappa \to \infty$ we can then replace the $R \times S^1$ string world sheet by the $R \times R$ one, i.e. i.e. the summation over the spatial momentum modes can be replaced by an integral [10, 15]. As a result, the dependence on κ in the effective action will factorize.[9]

We shall assume that the analytic continuation in the parameter m makes sense beyond the 1-loop level. We shall not try to justify the relation between the quantum corrections to the two backgrounds (1.5) and (1.8) or (1.10) step by step; instead, we will compute the partition function or the 2d energy near (1.10) in the formal limit $\kappa' \to 0$ as a function of (in general, complex) argument m and at the very end set $m = -i\kappa$ where $\kappa \to \infty$. We

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[9]The argument about factorization of κ dependence is strictly true only if all divergences cancel out. If, e.g., IR divergences survive one could get non-analytic $\kappa^2 \ln \kappa$ contributions. We expect them to cancel in the final result. In particular, the analytic continuation in the winding m eliminates the tachyonic instability of the circular solution making the 2d momentum integrals better defined in the IR.

shall expect that the final result gives us, as it happened at the tree and the 1-loop level, the information about the 2-loop correction to the energy of the folded string solution.[10]

Another technical remark is that instead of directly computing the quantum correction to the energy of our soliton solution using operator methods we shall compute the value of the quantum effective 1-PI action evaluated on the classical solution.[11] As a preparation for the 2-loop computation we are interested in, let us explain how one can get the same 1-loop correction as in [11, 15] by starting with the 1-loop effective action $\Gamma_1 = -\ln Z_1$ instead of the usual expression for 1-loop energy correction in terms of the sum over the characteristic frequencies $\sum_n \omega_n$.[12]

1.2 One-loop approximation

Let us first recall the expression for the leading term in the 1-loop correction to the energy of the folded string found in the scaling limit (1.6) with $\nu = 0$ ($\kappa \to \frac{1}{\pi} \ln S$) [11, 15]:[13]

$$E = \frac{1}{\kappa} E_{2d}, \qquad E_{2d}^{(1)} = \pi \kappa^2 a_1, \qquad a_1 = \frac{1}{\pi} \int_0^\infty dp \, \omega(p), \qquad (1.12)$$

$$\omega(p) = \sqrt{p^2 + 4} + 5\sqrt{p^2} + 2\sqrt{p^2 + 2} - 8\sqrt{p^2 + 1} \,. \qquad (1.13)$$

Here $\omega(p)$ contains the contributions of 8 bosonic and 8 fermionic fluctuation modes. The integral over p gives[14]

$$a_1 = -\frac{3\ln 2}{\pi} \,. \qquad (1.14)$$

We get the same result if we consider instead the expression for the Euclidean partition function and define E_{2d} as the effective action Γ divided over the 2-d time interval, i.e. at one loop

$$\Gamma_1 = -\ln Z_1 = V_2 \int \frac{d^2q}{(2\pi)^2} \, \mathcal{Z}_1(q^2), \qquad (1.15)$$

$$\mathcal{Z}_1(q^2) = \frac{1}{2} \left[\ln(q^2 + 4) + 5\ln q^2 + 2\ln(q^2 + 2) - 8\ln(q^2 + 1) \right] \,. \qquad (1.16)$$

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[10]This belief is based on the intuition that the string energy has a meaning when considered as a function of the complex values of its parameters, i.e. that different analytic continuations in parameters give values of the energy for different physical configurations. In short, having two classical solutions related by an analytic continuation in coordinates and parameters we shall assume that this relation holds also at the quantum level. We cannot of course consider the rotating solution as physical in the limit (1.11) (e.g., its energy is not defined if $\kappa' = 0$) but we shall assume that this limit of its energy defined for complex κ' and m has a meaning of the energy of the folded solution.

[11]Note that quantum corrections should not change the form of the classical solution due to its homogeneous nature. This case is similar to the case of a constant abelian gauge strength background in gauge theory.

[12]The two expressions are of course related in general by integrating out over p_0 component of the 2d momentum with the $i\epsilon$ prescription, but here in the absence of the UV divergences even a formal Euclidean continuation and direct integration over p_0 is enough to obtain the required result.

[13]Since $t = \kappa\tau$, the space-time energy is related [10] to the 2d energy by $E = \frac{1}{\kappa} E_{2d}$; in the limit $\kappa \to \infty$ the 2d energy E_{2d} scales as κ^2.

[14]For the reasons mentioned above, this integral happens to be essentially the same as in the case of the 1-loop correction to the energy of the circular $J_1 = J_2$ string solution in SU(2) sector [11, 27] considered in [17] and in appendix C of [29].

Here $V_2 = LT$ is the 2-d volume which factorises since our background is homogeneous: the fluctuation Lagrangian has constant coefficients and is thus translationally invariant. We assumed that the original coordinates τ and σ were rescaled by κ (this decompactifies the spatial direction in the limit $\kappa \to \infty$), so that

$$L = 2\pi\kappa\,, \qquad T = \kappa\bar{T}\,, \qquad V_2 = LT = 2\pi\kappa^2\bar{T}\,, \tag{1.17}$$

and thus

$$E_{2d}^{(1)} = \bar{T}^{-1}\Gamma_1\,, \qquad E_1 = T^{-1}\Gamma_1 = 2\pi\kappa \int \frac{d^2q}{(2\pi)^2}\ \mathcal{Z}_1(q^2)\,. \tag{1.18}$$

The integral over the 2d momentum is defined using the Euclidean continuation, i.e. $q^2 = q_0^2 + q_1^2$. Introducing the polar momentum space coordinates $d^2q = qdqd\phi$ and integrating over ϕ we end up with

$$E_{2d}^{(1)} = \frac{1}{2}\kappa^2 \int_0^\infty dv\ \mathcal{Z}_1(v)\,, \qquad\qquad v \equiv q^2\,. \tag{1.19}$$

This leads to the same expression for a_1 in $E_{2d}^{(1)} = \pi\kappa^2 a_1$ as in (1.12), (1.14).

1.3 Structure of the paper

Below we shall assume that a procedure of finding the quantum correction to the energy similar to the one described above at the 1-loop order should apply also at the 2-loop order and will therefore concentrate on the computation of the 2-loop correction to the 1-PI 2d effective action in the background (1.10) in the limit (1.11).

This is a technically involved computation. One issue is the large number of fields (10 bosonic and 32 fermionic) implying a large number of 2-loop Feynman graphs with non-diagonal propagators. Another is the presence of gauge symmetries — 2d diffeomorphisms (which we will fix by the conformal gauge) and the fermionic κ-symmetry. The preservation of the latter is expected to be quite subtle at higher loop orders. The complicated structure of the GS action makes the verification of cancellation of UV divergences (power-like, $\ln^2 \Lambda$ and $\ln \Lambda$ ones) non-trivial at the 2-loop order.[15]

We shall start in section 2 with determining the contribution of the 2-loop graphs containing the bosonic fluctuations. Section 2.1 will review some general facts about 2-loop renormalization of generic bosonic 2d sigma model in dimensional regularization, pointing out in particular that for symmetric spaces like $AdS_5 \times S^5$ the corresponding effective action does not contain $\ln^2 \Lambda \sim \frac{1}{\epsilon^2}$ UV divergences. In section 2.2 we shall present the form of the bosonic part of the $AdS_5 \times S^5$ action expanded to quartic order near the background (1.10), (1.11) and in section 2.3 will collect the expressions for the corresponding 2-loop

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[15]A crucial issue is that of an invariant UV regularization. Since the $AdS_5 \times S^5$ action contains the WZ-type term with $\epsilon^{\alpha\beta}$ tensor there are many analogies with 2-loop computations in bosonic sigma models with B_{mn} coupling (see, e.g., [37, 38, 43]). Other technical issues discussed below are cancellation of IR divergences (which would be automatically absent in the static gauge but formally may remain in the conformal gauge since some of the modes are massless) and the lack of manifest 2d Lorentz invariance ("spontaneously" broken beyond quadratic order by our choice of the background).

momentum integrals. The explicit results for the integrals will be presented in section 2.4. In addition to the standard 2-loop logarithmic divergence (that should be cancelled by the fermions) we shall find that the non-trivial finite part of the bosonic contribution to the 2-loop coefficient a_2 in (1.2) is given by a linear combination of the two transcendental constants — the Catalan's constant K and a similar combination $\tilde{K}$ of particular values of the trigamma function. The bosonic contribution (2.53) happens to be a factor of 1.42 different from the value of [32] in (1.3).

In section 3 we shall summarize the results of the computation of the 2-loop graphs involving the fermionic variables of the $AdS_5 \times S^5$ action of [4] (the action is reviewed in appendix A). We first consider the "covariant" κ-symmetry gauge $\theta^1 = k\theta^2$ where k is a real number. The relevant quartic part of the superstring action is given explicitly in appendix B. As we explain in appendix C, using a similar $k = 1$ gauge in the flat-space GS action one finds that the corresponding 2-loop graphs vanish in dimensional regularization, i.e. the 2-loop term in the flat-space partition function vanishes, in agreement with its triviality in the light-cone gauge.

Computing the corresponding 2-loop graphs resulting from vertices in the $AdS_5 \times S^5$ action (using a `Mathematica`-based program to evaluate several hundred Feynman diagrams) we found that their contribution to the effective action contains $\ln^2 \Lambda$ UV divergences. Since these were absent in the bosonic contribution, this contradicts the expected finiteness of the $AdS_5 \times S^5$ string. Moreover, the coefficients of both the divergent and the finite 2-loop part happen to depend on the gauge-fixing parameter k. This should not happen in an expansion near a classical solution and suggests a potential problem in our method of computation which we are unable to resolve at the moment.

For that reason we also redo the computation in a different κ-symmetry gauge $\Gamma_+ \theta^I = 0$ which is a direct analog of the usual light-cone gauge in flat space. The $AdS_5 \times S^5$ action in that gauge is presented in appendix D. There we show also that expanding the $AdS_5 \times S^5$ action near a null geodesic that wraps big circle of S^5 and computing the resulting 2-loop correction using $\Gamma_+ \theta^I = 0$ gauge one finds that it vanishes, in agreement with the BPS nature of the BMN vacuum state. Expanding near our background (1.10), (1.11) using the light-cone κ-symmetry gauge we find again that the 2-loop $\ln^2 \Lambda$ divergences do not cancel.[16]

An indication of consistency of our computation is that the non-trivial finite part of the 2-loop effective action is found to be the same in the $\Gamma_+ \theta^I = 0$ and in the $\theta^1 = \theta^2$ (i.e. $k = 1$) gauges. This finite part is proportional to the Catalan's constant. Thus, while we are currently unable to verify the 2-loop finiteness of the $AdS_5 \times S^5$ string action, an unambiguous conclusion of our work appears to be that the fermionic contribution to the 2-loop coefficient a_2 should be proportional to the transcendental Catalan's constant. We comment on the specific value of a_2 that follows from our light-cone gauge computation at

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[16]It is hard to attribute this lack of cancellation to a problem with the quartic fermion terms in the classical $AdS_5 \times S^5$ action as given in appendix A. Indeed, these terms provide the four-fermion entries of the tree-level scattering matrix which have been tested in [44, 45]. Moreover, these terms contribute nontrivially in the near BMN expansion, leading, as discussed in appendix D.1, to the expected cancellation of the 2-loop correction to string world-sheet partition function in the expansion near a null geodesic.

the end of section 3 and present some concluding remarks in section 4.

Some details of computation of 2-loop momentum integrals are discussed in appendix E.

2. Bosonic contribution to the 2-loop effective action

The bosonic part of the $AdS_5 \times S^5$ superstring action in the conformal gauge is simply the direct sum of the standard 2d sigma models on AdS_5 and S^5. The corresponding quantum theories are decoupled before fermions are switched on. Here we shall consider the 2-loop contributions of the bosonic fluctuations near the string background (1.10), (1.11).

2.1 General remarks on bosonic sigma model

The 2d sigma model action is (here we assume a Euclidean world-sheet signature)

$$I = \frac{1}{4\pi\alpha'} \int d^2\sigma \; G_{\mu\nu}(x) \; \partial^\alpha x^\mu \; \partial_\alpha x^\nu \, , \tag{2.1}$$

where in the case of our interest $G_{\mu\nu}$ is the metric of $AdS_5 \times S^5$ with radius a, $\sqrt{\lambda} = \frac{a^2}{\alpha'}$. If we use an explicit UV cutoff $\Lambda \to \infty$, the non-trivial power divergences in the partition function or in the effective action (computed by expanding near a solution of the classical equations of motion) should be cancelled by the covariant measure contribution in $Z = \int \prod_\sigma dx(\sigma) \sqrt{G(x(\sigma))} \; e^{-I[x]}$, i.e. by the contribution of the counterterm

$$\Delta I = -\frac{1}{2} \int d^2\sigma \; \mathrm{Tr} \ln G(x) \; \delta^{(2)}(\sigma, \sigma) \, , \qquad \delta^{(2)}(\sigma, \sigma) = \frac{1}{4\pi}\Lambda^2 \tag{2.2}$$

added to the bare action.[17]

If we use covariance-preserving dimensional regularization all power divergences will be absent automatically, i.e. the only potential divergences at the 2-loop level will be $\frac{1}{\epsilon} \sim \ln \Lambda$ and $\frac{1}{\epsilon^2} \sim \ln^2 \Lambda$ ones. As at the 1-loop level (1.16), the logarithmic divergences are expected to cancel at the end between the bosonic and fermionic contributions.

At the same time, it is easy to see that $\ln^2 \Lambda$ divergences should cancel separately for bosons (and thus also separately for the fermions). This follows from the basic renormalization properties of the sigma model in the case of the target-space metric corresponding to the Einstein space $R_{\mu\nu} = k G_{\mu\nu}$. Indeed, let us recall few basic facts about 2d sigma model renormalization in dimensional regularization (see, e.g., [47–49]). Using subscript 0 to denote bare quantities and μ for the renormalization scale we have for the partition function[18]

$$Z_0(G_0, \epsilon) = Z(G, \mu) \, , \qquad \mu \frac{\partial Z}{\partial \mu} + \beta \cdot \frac{\partial Z}{\partial G} = 0 \, . \tag{2.3}$$

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[17]This covariant measure factor may be understood as appearing from a 1-st order "phase-space" formulation upon integration over the momenta. More generally (in the bosonic sigma model context), the quadratic divergences may be absorbed into renormalization of the dimension 0 "tachyon" coupling, so in the bosonic string context the choice of the measure is like a choice of a bare value of the tachyon field (see, e.g., [46]).

[18]Since we will be expanding near a classical solution, we will not need to worry about field renormalization; the parameters of our background cannot get renormalized. As was already mentioned above, for a homogeneous solution there is also no reason to expect any change in the form of the background due to quantum corrections to the effective action.

Here $d = 2 - 2\epsilon$, $\frac{1}{\epsilon} \sim \ln \Lambda \to \infty$ and

$$G_0 = \mu^{-2\epsilon}\left[G + \frac{1}{\epsilon}T_1(G) + \frac{1}{\epsilon^2}T_2(G) + \cdots\right], \qquad (2.4)$$

so that from $\frac{dG_0}{d\ln\mu} = 0$ we get

$$\hat{\beta} = \frac{dG}{d\ln\mu} = 2\epsilon G + \beta, \qquad \beta = 2\left(1 - G\cdot\frac{\partial}{\partial G}\right)T_1, \qquad (2.5)$$

$$\left(1 - G\cdot\frac{\partial}{\partial G}\right)T_2 = \left(1 - G\cdot\frac{\partial}{\partial G}\right)T_1 \cdot \frac{\partial}{\partial G}T_1 . \qquad (2.6)$$

To the 2-loop order

$$(T_1)_{\mu\nu} = \frac{1}{2}\alpha' R_{\mu\nu} + \frac{1}{8}\alpha'^2 R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} + \cdots, \qquad (2.7)$$

$$(T_2)_{\mu\nu} = \frac{1}{16}\alpha'^2\left(D^\alpha D_\alpha R_{\mu\nu} - D^\alpha D_\nu R_{\mu\alpha} - D^\alpha D_\mu R_{\nu\alpha} + D_\mu D_\nu R\right), \qquad (2.8)$$

$$\beta_{\mu\nu} = \alpha' R_{\mu\nu} + \frac{1}{2}\alpha'^2 R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} + \cdots. \qquad (2.9)$$

In the case when the metric is the direct product of the AdS_5 and S^5 parts the Ricci tensor is covariantly constant so that for each factor $T_2 = 0$, i.e. there are no $\frac{1}{\epsilon^2} \sim \ln^2\Lambda$ divergences.[19]

For the S^N sigma model (with radius a playing the role of the running coupling constant) we have $R_{\mu\alpha\beta\gamma} = \frac{1}{a^2}(G_{\mu\beta}G_{\alpha\gamma} - G_{\mu\gamma}G_{\alpha\beta})$, $R_{\mu\nu} = \frac{(N-1)}{a^2}G_{\mu\nu}$ so that

$$(T_1)_{\mu\nu} = \frac{N-1}{2\sqrt{\lambda}}\left[1 + \frac{1}{2\sqrt{\lambda}} + O\left(\frac{1}{(\sqrt{\lambda})^2}\right)\right]G_{\mu\nu}, \qquad \sqrt{\lambda} = \frac{a^2}{\alpha'}. \qquad (2.10)$$

The corresponding 2-loop beta-function is of course the same as for the $O(N+1)$ sigma model [50, 51], i.e. (for $\alpha' = 1$) we get $\beta = \frac{da^2}{d\ln\mu} = (N-1)(1 + \frac{1}{a^2}) + \cdots$. For AdS_N one needs to invert the sign of the first term ($a^2 \to -a^2$).

The coefficients of the logarithmic divergences in the sigma model effective action computed in a particular background should be consistent with these general results. The divergent part of the effective action should be cancelled by the cutoff dependent terms in the bare sigma model action. Evaluated on the background (1.10) the latter is given by (for the S^5 part of the bosonic action, $N = 5$)

$$I_0 = \frac{\sqrt{\lambda}}{4\pi}\,\mu^{-2\epsilon}\left(1 + \frac{2}{\sqrt{\lambda}\,\epsilon} + \frac{1}{(\sqrt{\lambda})^2\epsilon} + \cdots\right)\int d\tau \int_0^{2\pi} d\sigma\,(m^2 - w^2)$$

$$= -\kappa^2\mu^{-2\epsilon}\left(\sqrt{\lambda} + \frac{2}{\epsilon} + \frac{1}{\sqrt{\lambda}\,\epsilon} + \cdots\right)\int d\tau, \qquad (2.11)$$

where we used that in the scaling limit $m^2 \to -w^2 = \kappa^2$ we get $m^2 - w^2 \to -2\kappa^2$.[20]

---

[19]The cancellation of $\ln^2$-divergences implies also the cancellation of $\ln$-divergences with transcendental coefficients like $\ln 2$ or Euler constant γ.

[20]The 1-loop coefficient here agrees with the UV divergent term coming from the bosonic part of the 1-loop effective action (1.16). Note that $\frac{1}{2}[\ln\det(-\partial^2 + M^2)]_\infty = \frac{1}{4\pi}V_2 M^2 \ln\frac{\Lambda}{\mu} = \frac{1}{8\pi\epsilon}V_2 M^2$.

2.2 $AdS_5 \times S^5$ sigma model fluctuation action

As a preparation for the 2-loop computation of the effective action let us now consider the $AdS_5 \times S^5$ bosonic action in conformal gauge expanded near the background (1.10), (1.11) to quartic order in fluctuation fields.

We shall adopt the following parametrization of the AdS_5 and S^5 parts of the metric

$$ds^2 = (ds^2)_{AdS_5} + (ds^2)_{S_5} \,, \tag{2.12}$$

$$(ds^2)_{AdS_5} = -\left(\frac{1 + \frac{1}{4}z^2}{1 - \frac{1}{4}z^2}\right)^2 dt^2 + \frac{dz^k dz_k}{(1 - \frac{1}{4}z^2)^2} \,, \qquad k = 1, 2, 3, 4 \,, \tag{2.13}$$

$$(ds^2)_{S_5} = \frac{dx^2 + dy^2 - (xdy - ydx)^2}{1 - x^2 - y^2} + (1 - x^2 - y^2) \times$$
$$\times \left(d\psi^2 + \cos^2 \psi \, d\phi_2^2 + \sin^2 \psi \, d\phi_3^3\right) \,. \tag{2.14}$$

The somewhat unusual form of the S^5 metric is chosen so that to have a regular expansion near the S^3 solution (1.10).[21] As discussed in section 1.1 above, we will be interested in the special case of the formal analytic continuation (1.9) of this solution with the parameters given by (1.11), i.e.

$$t = 0, \qquad z_k = 0, \qquad x = 0, \qquad y = 0 \,,$$
$$\psi = \frac{\pi}{4}, \qquad \phi_2 = \kappa(\tau - \sigma) \,, \qquad \phi_3 = \kappa(\tau + \sigma) \,, \tag{2.15}$$

Expanding the bosonic part of the string action to quartic order in fluctuations near this background

$$t = \tilde{t}, \qquad z_k = \tilde{z}_k \,, \qquad x = \tilde{x} \,, \qquad y = \tilde{y} \,, \qquad \psi = \frac{\pi}{4} + \tilde{\psi} \,,$$
$$\phi_2 = \kappa(\tau - i\sigma) + \tilde{\varphi}_2 - \tilde{\varphi}_3 \,, \qquad \phi_3 = \kappa(\tau + i\sigma) + \tilde{\varphi}_2 + \tilde{\varphi}_3 \,, \tag{2.16}$$

we get for the quadratic, cubic and quartic terms in the bosonic action

$$I_B = \int d\tau \int_0^{2\pi} d\sigma \, \mathcal{L}_B = -\sqrt{\lambda} \, \kappa^2 \int d\tau + \int d\tau \int_0^{2\pi} d\sigma \, (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \cdots) \,, \tag{2.17}$$

$$\mathcal{L}_2 = -\frac{\sqrt{\lambda}}{4\pi}\Big[- (\partial_\alpha \tilde{t})^2 + (\partial_\alpha \tilde{z}_k)^2 + (\partial_\alpha \tilde{x})^2 + (\partial_\alpha \tilde{y})^2 + 2\kappa^2 \tilde{x}^2 + 2\kappa^2 \tilde{y}^2$$
$$+ (\partial_\alpha \tilde{\psi})^2 + (\partial_\alpha \tilde{\varphi}_2)^2 + (\partial_\alpha \tilde{\varphi}_3)^2 + 4\kappa \, \tilde{\psi}(\partial_\tau \tilde{\varphi}_3 + i \, \partial_\sigma \tilde{\varphi}_2) \, \Big] \,, \tag{2.18}$$

$$\mathcal{L}_3 = -\frac{\sqrt{\lambda}}{4\pi}\Big[2\kappa \, (\tilde{x}^2 + \tilde{y}^2) \, (\partial_\tau \tilde{\varphi}_2 + i\partial_\sigma \tilde{\varphi}_3) - 4\tilde{\psi} \, \partial_\alpha \tilde{\varphi}_2 \partial^\alpha \tilde{\varphi}_3 \, \Big] \,, \tag{2.19}$$

$$\mathcal{L}_4 = \frac{\sqrt{\lambda}}{24\pi}\Big[\quad 3(\tilde{z}_k)^2 \left(-2(\partial_\alpha \tilde{t})^2 + (\partial_\alpha \tilde{z}_n)^2\right)$$
$$- 6 \left(\tilde{x}^2 + \tilde{y}^2\right) \left((\partial_\alpha \tilde{\varphi}_2)^2 + (\partial_\alpha \tilde{\varphi}_3)^2 + (\partial_\alpha \tilde{\psi})^2 + 4\kappa \, \tilde{\psi} \, (\partial_\tau \tilde{\varphi}_3 + i\partial_\sigma \tilde{\varphi}_2)\right)$$
$$+ 6 \left(\tilde{x}\partial_\alpha \tilde{x} + \tilde{y}\partial_\alpha \tilde{y}\right)^2 - 16\kappa \, \tilde{\psi}^3(\partial_\tau \tilde{\varphi}_3 + i\partial_\sigma \tilde{\varphi}_2) \Big] \,. \tag{2.20}$$

---

[21]The standard metric $(ds^2)_{S_5} = d\theta^2 + \cos^2 \theta \, d\phi_1^2 + \sin^2 \theta \, (d\psi^2 + \cos^2 \psi \, d\phi_2^2 + \sin^2 \psi \, d\phi_3^3)$ is related to the above one by the following coordinate transformation: $x = \cos\theta \cos\phi_1, \quad x = \cos\theta \sin\phi_1$.

Let us now make a few remarks.

Since the background values in (2.15), (2.16) depend on κ only in combination with world-sheet coordinates, we can factorize the κ-dependence in the Lagrangian ($\mathcal{L} \to \kappa^2 \mathcal{L}$) by making the rescaling

$$\kappa\tau \to \tau, \qquad \kappa\sigma \to \sigma \ .$$

This rescaling gives an equivalent theory assuming that scale invariance survives at the quantum level; this is not the case in the pure bosonic theory but should be so once fermions are added.

After the rescaling by κ (and assuming the cutoff dependence cancels out at the end) the string action on $R_\tau \times (S^1)_\sigma$ will depend on κ through the upper limit of integration $2\pi\kappa$ over rescaled σ. In the limit $\kappa \to \infty$ we are interested in we can then decompactify the spatial world-sheet dimension and thus use momentum representation with continuous spatial components.

The 1-loop correction to the effective action that follows from (2.18) can be easily seen to be in agreement with the bosonic part of (1.15), (1.16). The quadratic part of the fluctuation action (2.18) can be diagonalized by a (non-local) "rotation" of the three S^3 fields (see [27]). This will bring in one massive and two massless modes in the $(\tilde{\psi}, \tilde{\varphi}_2, \tilde{\varphi}_3)$ sector. The resulting quadratic fluctuation part of the superstring action will have the form of 2d Lorentz invariant collection of massive bosonic and fermionic fields, but higher-order terms in fluctuations will no longer have 2d Lorentz invariance (which is "spontaneously broken" by our choice of the background). Expressed in terms of the "rotated" fields the interaction terms will have non-local form. For that reason here we choose not to perform this diagonalization explicitly and use non-diagonal propagator instead.

As was already mentioned, in conformal gauge the bosonic contributions of AdS_5 and S^5 parts factorize. If we formally set $\kappa = 0$ in (2.18), (2.19), (2.20), i.e. consider the case of trivial background in all directions, then the AdS_5 and S^5 contributions to the partition function will become similar.[22]

In the action (2.17) we assumed the Minkowski world-sheet signature $(-, +)$; the action is not real because of our choice of the imaginary value of the winding parameter m. The Euclidean action obtained by continuing $\tau \to i\tau$ is also not real but the imaginary parts are linear in κ and derivatives, so the partition function and the effective action will be real. We shall continue to Euclidean signature at the level of momentum-space integrals.

2.3 Structure of 2-loop quantum corrections

The 2-loop contributions to the effective action in a theory like (2.17) with three-point and four-point vertices is given by the Feynman diagrams of the two topologies shown in figure 1. In general, the lines in these diagrams may be either bosons or fermions. The 2-loop 1-PI effective action is then given by

$$\Gamma = V_2 \bar{\Gamma}, \qquad \bar{\Gamma} = \bar{\Gamma}_1 + \bar{\Gamma}_2 + \cdots, \qquad \bar{\Gamma}_2 = \bar{\Gamma}_{\text{cubic}} + \bar{\Gamma}_{\text{quartic}} + \delta\bar{\Gamma}_{\text{measure}} \ . \quad (2.21)$$

---

[22]The fact that in the AdS_5 part we have only quartic interaction while in the S^5 part we also have a cubic one is an artifact of a particular parametrization and the choice of the expansion point used.

Figure 1: Two-loop contributions (momentum conservation $q_i + q_j + q_k = 0$ is assumed).

Here V_2 is the volume factor as in (1.15) (our background is homogeneous), i.e. $\bar{\Gamma}$ stands for the effective Lagrangian. $\delta\bar{\Gamma}_{\text{measure}}$ is the contribution coming from the measure counterterm (2.2) expanded to quadratic order in fluctuations, i.e. (after Wick rotation)

$$\delta\mathcal{L} = -\frac{1}{2}\big[3\tilde{z}^2 - 2\tilde{x}^2 - 2\tilde{y}^2 - 4\tilde{\psi}^2 + O(\tilde{\phi}^3)\big]\int \frac{d^d q_j}{(2\pi)^d \mu^{d-2}}\,, \tag{2.22}$$

where $\int \frac{d^d q_j}{(2\pi)^d \mu^{d-2}}$ is the (correctly normalized) integral representation of $\delta^{(2)}(0)$. The insertion of this counterterm into a 1-loop diagram will cancel all quadratic divergences in the 2-loop effective action. We will be using the dimensional regularization with μ as a renormalization scale and $d = 2 - 2\epsilon$ since this is an invariant regularization preserving the symmetries of the sigma model. Power divergences can be ignored in dimensional regularization but it is sometimes useful to track their cancellation against the measure as a check of combinatorial factors.

To compute the 2-loop diagrams we need to work out the propagator. The quadratic terms in (2.18) contain off-diagonal mixings which can be readily diagonalized as in [27]. However, we found it more convenient to keep the propagator off-diagonal. Ordering the fluctuation fields as follows

$$\Phi_i = \{\tilde{t}; (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4); (\tilde{x}, \tilde{y}); (\tilde{\psi}, \tilde{\varphi}_2, \tilde{\varphi}_3)\} \tag{2.23}$$

one finds from (2.18)

$$\Delta^{-1}(q) = \frac{2\pi}{\sqrt{\lambda}}\begin{pmatrix} -\frac{1}{q^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{q^2}\mathbb{1}_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q^2+2}\mathbb{1}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{q^2+4} & \frac{2q_1}{q^2(q^2+4)} & \frac{-2i\,q_0}{q^2(q^2+4)} \\ 0 & 0 & 0 & -\frac{2q_1}{q^2(q^2+4)} & \frac{(q^2)^2-4q_0^2}{(q^2)^2(q^2+4)} & \frac{2i\,q_0 q_1}{(q^2)^2(q^2+4)} \\ 0 & 0 & 0 & \frac{2i\,q_0}{q^2(q^2+4)} & \frac{2i\,q_0 q_1}{(q^2)^2(q^2+4)} & \frac{(q^2)^2+4q_1^2}{(q^2)^2(k^2+4)} \end{pmatrix} \tag{2.24}$$

Here $q_\alpha = (q_0, q_1)$ is 2-momentum. We have rescaled the coordinates by κ (with $\kappa \to \infty$) and will assume that momenta take continuous values. Continuation to Euclidean signature is done by $q_0 \to -iq_0$. This eliminates i-factors from the propagator. The 1-loop effective action is then $\Gamma_1 = \frac{1}{2}\text{Tr}\ln\Delta$ and agrees with (1.15), (1.16).[23]

---

[23]We may formally ignore the "ghost" nature of the $\tilde{t}$ fluctuation and then the 1-loop contribution of

Defining the cubic vertex as $V_{ijk}(q_i, q_j, q_k) = \frac{\partial^3 \mathcal{L}}{\partial \Phi_i \partial \Phi_j \partial \Phi_k}\big|_{\Phi=0}$, i.e. writing the (Euclidean) fluctuation Lagrangian corresponding to (2.17) as

$$\mathcal{L} = \frac{1}{2}\Phi_i \Delta_{ij}\Phi_j + \frac{1}{3!}V_{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{4!}V_{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l + \cdots, \tag{2.25}$$

we can compute the contribution of the graphs with topology (a) in figure 1 as

$$
\begin{aligned}
\bar{\Gamma}_{\text{cubic}} &= c_3 \int \frac{d^d q_i d^d q_j}{(2\pi)^{2d}\mu^{2d-4}} \, V_{ijk}V_{i'j'k'}\Delta_{ii'}^{-1}\Delta_{jj'}^{-1}\Delta_{kk'} \\
&= c_3 \frac{4\pi}{\sqrt{\lambda}} \int \frac{d^d q_i d^d q_j}{(2\pi)^{2d}\mu^{2d-4}} (\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_N)
\end{aligned}
\tag{2.26}
$$

where[24]

$$c_3 = -\frac{1}{12}$$

is the combinatorial factor of the diagram and we have solved the vertex momentum conservation constraint by setting $q_k = -(q_i + q_j)$. We assume that continuation to d dimensions is done at the level of the momentum integrals, and μ-factors are introduced to balance the dimensions. The overall factor of κ^2 is included in the volume V_2 in (2.21) as in (1.15), (1.17).

There are many equivalent expressions for the integrands $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_N$; the one which exposes both the UV and IR convergence properties of the loop integrals is:[25]

$$\mathcal{I}_1 = 3\frac{4}{q_i^2 + 4}, \tag{2.27}$$

$$\mathcal{I}_2 = 3\left[-\frac{2}{q_i^2 q_j^2} + \frac{4}{q_i^2(q_j^2+4)} - \frac{4}{(q_i^2+2)(q_j^2+2)} - \frac{14 + \frac{8}{d}}{(q_i^2+4)(q_j^2+4)} \right], \tag{2.28}$$

$$\mathcal{I}_3 = 3\left[\frac{8}{q_i^2 q_j^2} + \frac{16}{(q_i^2+2)(q_j^2+2)} - \frac{8}{(q_i^2+4)(q_j^2+4)} \right]\frac{1}{q_k^2+4}, \tag{2.29}$$

$$
\begin{aligned}
\mathcal{I}_N = 3\Bigg[&(q_{i1}q_{j0} + q_{i0}q_{j1})^2 \frac{8(q_i^2 + q_j^2 - q_k^2)^2}{(q_i^2)^2(q_i^2+4)(q_j^2)^2(q_j^2+4)(q_k^2+4)} \\
&- (q_{i0}^2 - q_{i1}^2)^2 \frac{16}{(q_i^2)^2(q_i^2+4)(q_j^2+2)(q_k^2+2)} \\
&- (q_{i0}q_{j0} - q_{i1}q_{j1})(q_{i0}q_{k0} - q_{i1}q_{k1})\frac{16\left[(q_i^2)^2 - (q_j^2-q_k^2)^2\right]}{(q_i^2)^2(q_i^2+4)q_j^2(q_j^2+4)q_k^2(q_k^2+4)} \Bigg]
\end{aligned}
\tag{2.30}
$$

two massless "longitudinal" modes is cancelled by the conformal gauge ghost contribution to the partition function. The "ghost" sign of the time direction is irrelevant also for the higher-loop corrections: since time direction enters the action only quadratically, it can be integrated out once and for all (e.g., with $t \to it$ prescription to make Euclidean path integral convergent) and that does not lead to any sign changes compared to the case when t would have "physical" sign.

[24]Here we assume Euclidean continuation, i.e. $e^{-\Gamma} = \int [d\Phi]\, e^{-S}$, $S = \int d^2\sigma \mathcal{L}$.

[25]Here q_i and q_j denote two momenta without any summation over i, j and $q_k = -(q_i + q_j)$.

In (2.28) $d = 2 - 2\epsilon$.[26] We also continued to Euclidian space by replacing $q_{j0} \to -iq_{j0}$, so that in the above expressions $q_j^2 = q_{j0}^2 + q_{j1}^2$.

$\mathcal{I}_1$ and $\mathcal{I}_2$ give rise to UV-divergent integrals; the integral of $\mathcal{I}_1$ contains power-like divergences and the integral of $\mathcal{I}_2$ – logarithmic divergences. The first two terms in $\mathcal{I}_2$ and the first term in $\mathcal{I}_3$ give rise to IR-divergent integrals. In addition to the dimensional regularization for the UV divergences we shall introduce a small mass parameter m_0 to regularize the IR divergences.[27]

The subscript N on $\mathcal{I}_N$ is used to indicate that this integrand does not look 2d Lorentz invariant. However, the integral of $\mathcal{I}_N$ (which is UV and IR finite) can be expressed in terms of Lorentz-invariant integrals. While the original sigma model action (the string action in conformal gauge) is 2d Lorentz-invariant, this symmetry is spontaneously broken by a choice of the background in (2.15), (2.16), i.e. (cf. (2.14))

$$\cos\psi d\phi_2 = N_\alpha d\sigma^\alpha, \quad \sin\psi d\phi_3 = N_\alpha^* d\sigma^\alpha, \quad N_\alpha = \frac{\kappa}{\sqrt{2}}(1, -i), \quad N_\alpha^* = \frac{\kappa}{\sqrt{2}}(1, i). \quad (2.31)$$

The 2-loop effective action then depends on the background through the mass terms (proportional to $N_\alpha^* N^\alpha = -\kappa^2$, etc.) and also through the explicit factors of N_α and N_α^* in the denominators of momentum integrals. Indeed, $\mathcal{I}_N$ in (2.30) is proportional to 4 factors of these vectors. Since the rest of the momentum integrands are Lorentz-covariant, they can be reduced to products of contractions between N_α and N_α^* factors and scalar Lorentz-invariant momentum integrals. We shall illustrate how that happens below. As a result, the corresponding term in Γ_2 will contain 4 factors of first derivatives of the background fields, i.e. will be proportional to $\partial^\alpha\phi_2\partial_\alpha\phi_2\partial^\beta\phi_3\partial_\beta\phi_3 + \cdots$ with coefficients that are given by Lorentz-invariant momentum integrals.[28]

Similarly, for the contribution of the quartic vertex in (2.25) $V_{ijkl} = \frac{\partial^4\mathcal{L}}{\partial\Phi_i\partial\Phi_j\partial\Phi_k\partial\Phi_l}\big|_{\Phi=0}$ to the diagram (b) in figure 1 we find

$$\bar{\Gamma}_{\text{quartic}} = c_4 \int \frac{d^d q_i d^d q_j}{(2\pi)^{2d}\mu^{2d-4}} V_{ijkl}\Delta_{ij}^{-1}\Delta_{kl}^{-1} = c_4 \frac{4\pi}{\sqrt{\lambda}} \int \frac{d^d q_i d^d q_j}{(2\pi)^{2d}\mu^{2d-4}} (\mathcal{J}_1 + \mathcal{J}_2), \quad (2.32)$$

where

$$c_4 = \frac{1}{8}$$

is the combinatorial factor. Despite the relatively complicated-looking quartic Lagrangian (2.20) the integrands $\mathcal{J}_1$ and $\mathcal{J}_2$ are very simple:

$$\mathcal{J}_1 = \frac{24}{q_i^2} - \frac{8}{q_i^2 + 2}, \quad (2.33)$$

$$\mathcal{J}_2 = -\frac{8}{(q_i^2 + 2)(q_j^2 + 2)} - \frac{32}{(q_i^2 + 4)(q_j^2 + 4)}. \quad (2.34)$$

---

[26]The factor $\frac{1}{d}$ in (2.28) came from a reduction of a tensor integral to a scalar integral due to symmetric integration: $\int \frac{d^d q_i d^d q_j}{(2\pi)^{2d}} \frac{(q_i \cdot q_j)^2}{(q_i^2+4)(q_j^2+4)} = d^{-1}\int \frac{d^d q_i d^d q_j}{(2\pi)^{2d}} \frac{q_i^2 q_j^2}{(q_i^2+4)(q_j^2+4)}$. In general, $\int \frac{d^d q}{(2\pi)^d} \frac{q^\alpha q^\beta}{(q^2+4)^n} = d^{-1}\int \frac{d^d q}{(2\pi)^d} \frac{\eta^{\alpha\beta} q^2}{(q^2+4)^n}$.

[27]We will not use regulators in finite integrals.

[28]Let us note that the use of dimensional regularization in a situation with Lorentz invariance spontaneously broken by either the background or by gauge choice is not uncommon (cf., e.g., discussions of YM theory in lightcone gauge [52]).

Both $\mathcal{J}_1$ and $\mathcal{J}_2$ lead to UV-divergent integrals – power-like and logarithmic, respectively. The contribution of the measure counterterm (2.22) is

$$\delta\bar{\Gamma}_{\text{measure}} = -\frac{4\pi}{\sqrt{\lambda}} \int \frac{d^d q_i d^d q_j}{(2\pi)^{2d}\mu^{2d-4}} \left(\frac{3}{q_i^2} - \frac{1}{q_i^2+2} - \frac{1}{q_i^2+4}\right). \quad (2.35)$$

It is not hard to check that it cancels all power-like divergences in the 2-loop integrals in (2.26) and (2.32). In particular, it cancels the contribution of the $\mathcal{J}_1$ integral in (2.33).

Let us note that if we formally consider the theory (2.17) defined on $R \times R$ and set $\kappa = 0$ then the corresponding 2-loop effective action will be given by (2.21) with (2.26) containing only "massless" limit $\frac{12}{q_i^2}$ of $\mathcal{I}_1$ in (2.27) and with (2.32) containing only the "massless" limit $\frac{16}{q_i^2}$ of $\mathcal{J}_1$ in (2.33). Their sum is then cancelled by the "massless" limit of the measure contribution (2.35) (with the integrand $\frac{1}{q_i^2}$). Thus $\Gamma_2(\kappa \to 0) \to 0$.

2.4 Evaluation of 2-loop momentum integrals

Combining the above 2-loop contributions we get for (2.21)

$$\begin{aligned}\bar{\Gamma}_2 &= \bar{\Gamma}_{\text{cubic}} + \bar{\Gamma}_{\text{quartic}} + \delta\bar{\Gamma}_{\text{measure}} \\ &= \frac{4\pi}{\sqrt{\lambda}} \int \frac{d^d q_i d^d q_j}{(2\pi)^{2d}\mu^{2d-4}} \left[\left(-\frac{1}{12}\mathcal{I}_2 + \frac{1}{8}\mathcal{J}_2\right) - \frac{1}{12}(\mathcal{I}_3 + \mathcal{I}_N)\right]. \end{aligned} \quad (2.36)$$

Here the contribution of the first parenthesis contains all UV divergences. It turns out that the contributions of states with mass-squared equal to 2 cancel between the topologies (a) and (b). Then we get $(d = 2 - 2\epsilon)$

$$-\frac{1}{12}\mathcal{I}_2 + \frac{1}{8}\mathcal{J}_2 = \frac{\epsilon}{1-\epsilon}\frac{1}{(q_i^2+4)(q_j^2+4)} + \frac{1}{2}\left(\frac{1}{q_i^2} - \frac{1}{q_i^2+4}\right)\left(\frac{1}{q_j^2} - \frac{1}{q_j^2+4}\right). \quad (2.37)$$

The contribution of the second term in (2.37) is UV-finite but IR-divergent. As was mentioned above, we shall regularize this IR divergence by introducing a small mass m_0. Using the standard integral

$$\begin{aligned}I(M^2) &\equiv \mu^{2\epsilon} \int \frac{d^d q}{(2\pi)^d}\frac{1}{q^2+M^2} = \frac{1}{(4\pi)^{d/2}}\frac{\pi}{\Gamma(2-\epsilon)\sin(\pi\epsilon)}\left(\frac{\mu^2}{M^2}\right)^\epsilon \\ &\approx \frac{1}{4\pi}\left[\frac{1}{\epsilon} + 1 - \gamma + \ln\frac{4\pi\mu^2}{M^2} + O(\epsilon)\right], \end{aligned} \quad (2.38)$$

we then find[29]

$$\bar{\Gamma}_2 = \frac{1}{4\pi\sqrt{\lambda}}\left[\frac{1}{\epsilon} + 3 - 2\gamma + 2\ln(\pi\mu^2) + \frac{1}{2}\ln^2\left(\frac{m_0^2}{4}\right)\right] - \frac{4\pi}{12\sqrt{\lambda}}\int\frac{d^2 q_i d^2 q_j}{(2\pi)^4}(\mathcal{I}_3 + \mathcal{I}_N). \quad (2.39)$$

---

[29]Here $\gamma = -\Psi(1) = 0.5772\ldots$ is the Euler constant. Let us also recall that we have rescaled the worldsheet variables by κ. If we did not do this but still formally decompactified the spatial direction of the world sheet we would get the first term here as

$$\bar{\Gamma}_2 = \frac{\kappa^2}{4\pi\sqrt{\lambda}}\left[\frac{1}{\epsilon} + 3 - 2\gamma + 2\ln\frac{\pi\mu^2}{\kappa^2} + \frac{1}{2}\ln^2\left(\frac{m_0^2}{4\kappa^2}\right)\right] + \text{finite}.$$

As expected for a symmetric-space sigma model, the double-pole $\frac{1}{\epsilon^2}$ UV divergences cancelled out (cf. (2.8), (2.11)). The effective action is found by multiplication of this expression by $V_2 = 2\pi\kappa^2\bar{T}$ as in (1.17).

Next, let us compute the integral of $\mathcal{I}_3$ in (2.29), (2.36), writing it as

$$I_3 = \int \frac{d^2q_i d^2q_j}{(2\pi)^4} \left(\mathcal{I}_{3,1} + \mathcal{I}_{3,2} + \mathcal{I}_{3,3}\right) = I_{3,1} + I_{3,2} + I_{3,3} \ . \tag{2.40}$$

The integral of the first term

$$I_{3,1} = \int \frac{d^2q_i d^2q_j}{(2\pi)^4} \frac{24}{q_i^2 q_j^2 ((q_i + q_j)^2 + 4)} \tag{2.41}$$

with two massless propagators is IR divergent and we need to regularize it by $m_0 \to 0$. This leads to an integral which is a special case of the following integral with 3 massive propagators with at least two equal masses[30]

$$I(M, M') = \int \frac{d^2q_i d^2q_j}{(2\pi)^4} \frac{1}{(q_i^2 + M^2)(q_j^2 + M'^2)[(q_i + q_j)^2 + M'^2]} \tag{2.42}$$

The calculation of this integral is standard: we Feynman-parametrize the propagators with equal masses and do the integral over q_j with the result:

$$I(M, M') = \frac{1}{4\pi} \int_0^1 dx \int \frac{d^2q_i}{(2\pi)^2} \frac{1}{(q_i^2 + M^2)[x(1-x)q_i^2 + M'^2]} \tag{2.43}$$

There is no need of Feynman parametrization for the second momentum integral; computing it directly leads to

$$I(M, M') = \frac{1}{(4\pi)^2} \int_0^1 dx \, \frac{\ln \frac{M^2}{M'^2} + \ln[x(1-x)]}{x(1-x)M^2 - M'^2} \ . \tag{2.44}$$

For generic values of M and M' the remaining integral leads to a hypergeometric function. However, (2.41) corresponds to $M = 2$, $M' = m_0 \to 0$. Expanding (2.44) in $M' = m_0 \to 0$ we get for (2.41)

$$(I_{3,1})_{m_0 \to 0} = \frac{6}{(4\pi)^2} \left[\frac{13}{3}\pi^2 + \ln^2\left(\frac{m_0^2}{4}\right)\right] \ . \tag{2.45}$$

Multiplying this by the $-\frac{1}{12}\frac{4\pi}{\sqrt{\lambda}}$ factor in (2.39) we conclude that the IR divergence from $I_{3,1}$ cancels the one in (2.39), so that the bosonic part of the effective action is IR finite.[31]

For the second term $\mathcal{I}_{3,2}$ we need (2.42) with $M^2 = 4$ and $M'^2 = 2$, with (2.44) then giving

$$\begin{aligned}
I_{3,2} &= \int \frac{d^2q_i d^2q_j}{(2\pi)^4} \frac{48}{(q_i^2 + 4)(q_j^2 + 2)((q_i + q_j)^2 + 2)} \\
&= \frac{24}{(4\pi)^2} \int_0^1 dx \frac{\ln[2x(1-x)]}{2x(1-x) - 1} = \frac{48}{(4\pi)^2}\mathrm{K} \ .
\end{aligned} \tag{2.46}$$

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[30]We may solve the momentum conservation condition as $q_k = -(q_i + q_j)$ or as $q_j = -(q_k + q_i)$; the final result is the same.

[31]This is of course what one should have expected since we are computing a physical quantity: the value of the (global symmetry invariant) effective action on a classical solution, cf. [53, 43].

Here K is the Catalan's constant,

$$\mathrm{K} \equiv \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \frac{1}{16}\left[\Psi'\left(\frac{1}{4}\right) - \Psi'\left(\frac{3}{4}\right)\right] = 0.915966\ldots, \tag{2.47}$$

where

$$\Psi'(z) = \psi_1(z) \equiv \frac{d^2}{dz^2}\ln\Gamma(z) \tag{2.48}$$

is the trigamma function.[32] For the third term in $\mathcal{I}_3$ in (2.29), (2.36) we need (2.42) with $M^2 = M'^2 = 4$ so that

$$\begin{aligned} I_{3,3} &= -\int \frac{d^2q_i d^2q_j}{(2\pi)^4} \frac{8}{(q_i^2+4)(q_j^2+4)((q_i+q_j)^2+4)} \\ &= -\frac{2}{(4\pi)^2}\int_0^1 dx \frac{\ln[x(1-x)]}{x(1-x)-1} = -\frac{8}{(4\pi)^2}\tilde{\mathrm{K}}, \end{aligned} \tag{2.49}$$

where

$$\tilde{\mathrm{K}} \equiv \frac{1}{72}\left[\Psi'\left(\frac{1}{6}\right) + \Psi'\left(\frac{1}{3}\right) - \Psi'\left(\frac{2}{3}\right) - \Psi'\left(\frac{5}{6}\right)\right] = 0.585976\ldots. \tag{2.50}$$

Let us note also an alternative representation for $\tilde{\mathrm{K}}$ similar to the one for K in (2.47) which follows from the series representation for $\Psi'(z)$[33]

$$\tilde{\mathrm{K}} = \frac{1}{2}\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)^2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)^2}\right]. \tag{2.51}$$

The calculation of the integral of $\mathcal{I}_N$ in (2.30) is described in appendix E.

Combining the expressions (2.45), (2.46), (2.49) and (E.18) we finally find for the bosonic contribution (2.39) to the 2-loop effective Lagrangian[34]

$$\bar{\Gamma}_{2\mathrm{B}} = \frac{1}{4\pi\sqrt{\lambda}}\left(\left[\frac{1}{\epsilon} + 3 - 2\gamma + 2\ln(\pi\mu^2)\right] - \left(2\mathrm{K} + \frac{4}{3}\tilde{\mathrm{K}}\right)\right). \tag{2.52}$$

The fermionic contribution is expected to cancel the divergent part and the associated finite terms, i.e. the first bracket in (2.52). The non-trivial finite bosonic contribution to the 2-loop string coefficient a_2 of $\ln S$ in (1.1), (1.2) should then be proportional to $2\mathrm{K} + \frac{4}{3}\tilde{\mathrm{K}}$: it is found as in (1.17), (1.18) by multiplying (2.52) by $2\pi\kappa \approx 2\ln S$. This gives

$$a_{2B} = -\frac{1}{\pi}\left(\mathrm{K} + \frac{2}{3}\tilde{\mathrm{K}}\right) \approx -0.41591. \tag{2.53}$$

This should be supplemented by the corresponding contribution of the 2-loop graphs with the fermions.

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[32] It admits the following series representation $\psi_1(z) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}$ and also satisfies a reflection formula $\psi_1(1-z) + \psi_1(z) = \pi^2\csc^2(\pi z)$. Note also that $\psi_1\left(\frac{1}{4}\right) = \pi^2 + 8\mathrm{K}$.

[33] We thank M. Staudacher for mentioning this representation to us and for emphasizing that K and $\tilde{\mathrm{K}}$ have the same "transcendentality" (cf. [54]).

[34] The rational term in the finite part of (2.39) cancelled out between I_3 and I_N.

3. Fermionic contribution to the 2-loop effective action

Let us now turn to the contribution to the 2-loop effective action coming from diagrams containing fermionic propagators. The relevant terms in the $AdS_5 \times S^5$ Lagrangian expanded near the background (2.15) can be symbolically written as

$$\mathcal{L}_F = \frac{1}{2}\theta K\theta + (\theta M_1\theta)Y_1\Phi + (\theta M_2\theta)\Phi Y_2\Phi + (\theta M_3\theta)(\theta M_4\theta) . \tag{3.1}$$

Here Φ stands for the bosonic fluctuation fields (2.23) and K, M_n, Y_k are combinations of Dirac matrices, numerical tensors and world sheet derivatives of the form $A + B^\alpha\partial_\alpha$. Their explicit form follows directly from the relations given in appendix A but are rather lengthy so we will not give it explicitly here.

As was already mentioned in the Introduction, because the fermionic kinetic term is only linear in derivative while the interaction vertices contain up to two derivatives, the GS string theory is formally of non-renormalizable type; this will manifest itself in the presence of higher power divergences.

Assuming the theory is actually finite, all of power divergences are expected to be cancelled by the contributions of the path integral measure and κ-symmetry ghosts (see appendix C for a discussion of this in the flat space case). Alternatively, one may choose to use dimensional regularization in which all power divergences are automatically set to zero. Then the remaining $\ln^2\Lambda \sim \frac{1}{\epsilon^2}$ divergences should cancel separately in the fermionic sector while the $\ln\Lambda \sim \frac{1}{\epsilon}$ contributions should cancel against the bosonic divergence in (2.52).

There are several potential ambiguities in how one deals with divergent integrals. Since the GS action contains a WZ type term with $\epsilon^{\alpha\beta}$ tensor, this creates a potential problem with direct application of dimensional regularization.[35] We shall assume that the dimensional regularization is applied only to scalar integrals at the last stage (after all power-divergent parts of the momentum integrands are separated), i.e. that all tensor algebra is done in $d = 2$; in particular, we shall assume that $\epsilon^{\alpha\beta}$ is not continued away from $d = 2$.[36]

Our assumption will be that such a restricted dimensional regularization prescription is consistent with the basic κ-symmetry of the theory at the quantum level. This is by no means obvious and a problem with κ-symmetry gauge dependence of the 2-loop result that we will encounter below appears to be an indication of a problem with this prescription.[37]

One natural choice of the κ-symmetry gauge (used at the one loop order in [9, 10]) is $\theta^1 = \theta^2$. This gauge is possible in type IIB string action where both Majorana-Weyl

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[35]Let us note also that the parameters of the κ-symmetry transformations are 2d self-dual vectors.

[36]This is somewhat different from the case of the bosonic sigma model with an antisymmetric tensor coupling [37, 38] where one could assume that $\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} = f(d)(\eta^{\alpha\gamma}\eta^{\beta\delta} - \eta^{\alpha\delta}\eta^{\beta\gamma})$ where $f(d) = 1 + a(d-2) + \cdots$, and then show that a regularization scheme ambiguity related to the choice of the coefficient a can be absorbed into a redefinition of the sigma model coupling parameters.

[37]The standard proof of gauge-independence of on-shell effective action assumes that gauge symmetry in question is preserved at the quantum level, i.e. implicitly assumes the existence of an invariant regularization (but the power counting renormalizability of the theory is of course not required).

fermions in the GS action have the same chirality. One of its advantages is preservation of global bosonic symmetries of the action. More generally, we may consider the gauge $\theta^1 = k\theta^2$ where k is a real parameter (see appendix B). Cancellation of k-dependence in the resulting effective action, i.e. its gauge-choice independence, would be a check of consistency of our computation procedure (in particular, of the regularization we use).

Let us first comment on the structure of the fermionic 2-loop contributions in the simpler case of $k = 1$ gauge. The quadratic part of the gauge-fixed action follows from (B.3), (B.4) and is given by

$$\mathcal{L}_{F2} = \sqrt{2}\kappa\bar{\theta}\left[\Gamma_8(\partial_\sigma - i\partial_\tau) - \Gamma_9(\partial_\sigma + i\partial_\tau)\right]\theta + 2i\kappa^2\bar{\theta}\Gamma_*\Gamma_8\Gamma_9\theta \equiv \frac{1}{2}\theta^T K\theta \ . \tag{3.2}$$

This leads to the propagator (where we again rescaled the momentum by κ)

$$K^{-1}(q) = \frac{1}{4\sqrt{2}(q^2 + 1)}\left[\Gamma_8(q_0 + iq_1) + \Gamma_9(q_0 - iq_1) - i\sqrt{2}\Gamma_*\Gamma_8\Gamma_9\right]\mathcal{C} \ . \tag{3.3}$$

As a result, all fermionic modes have mass equal to 1, while the bosonic modes in (2.24) had masses equal to 0, $\sqrt{2}$ and 2 (cf. the corresponding 1-loop expression in (1.16)).

There are 3 different types of 2-loop diagrams involving the fermions (see (3.1)):

(i) diagram in figure 1 (a) with two fermionic and one bosonic propagators (we shall call it "FFB" since it originates from the Yukawa interaction in (3.1));

(ii) diagram in figure 1 (b) with one bosonic and one fermionic propagators (originating from the "FFBB" interaction);

(iii) diagram in figure 1 (b) with two fermionic propagators (coming from "FFFF" vertex).

The most non-trivial contribution with the integrand containing *two* fermionic and *one* bosonic propagator may come only from the FFB diagram. Thus on general grounds we may expect that the finite part of the fermionic contribution which should supplement the finite bosonic contributions in (2.46) and (2.49) should be given by a combination of two possible finite integrals of the general form (2.42):[38]

$$I(\sqrt{2}, 1) = \int \frac{d^2q_i d^2q_j}{(2\pi)^4} \frac{1}{(q_i^2 + 1)(q_j^2 + 1)[(q_i + q_j)^2 + 2]} = \frac{1}{8\pi^2}\mathrm{K} \ , \tag{3.4}$$

$$I(2, 1) = \int \frac{d^2q_i d^2q_j}{(2\pi)^4} \frac{1}{(q_i^2 + 1)(q_j^2 + 1)[(q_i + q_j)^2 + 4]} = \frac{\ln 2}{8\pi^2} \ , \tag{3.5}$$

where in computing the integrals we used (2.44) and K is again the Catalan's constant as in (2.47).

It turns out that only $I(\sqrt{2}, 1)$ in (3.4) appears as a result of the actual computation of the FFB graph. This leads to the conclusion that the finite fermionic contribution can

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[38]The third possible integral $I(0, 1) = \int \frac{d^2q_i d^2q_j}{(2\pi)^4} \frac{1}{(q_i^2+1)(q_j^2+1)(q_i+q_j)^2}$ is IR divergent (cf. (2.45)) and does not give a non-trivial transcendental contribution to the finite part. It does not actually appear in the result of the computation.

alter the coefficient of the G-term in (2.52), (2.53). Assuming all other possible finite contributions like $\ln 2$ which accompany logarithmic divergences should cancel out, we are then led to the following prototype of the final answer for the coefficient a_2 in (1.2) (cf. (2.52), (2.53))

$$a_2 = a_{2B} + a_{2F} = -\frac{1}{\pi}\left[(1 + c_F)\mathrm{K} + \frac{2}{3}\tilde{\mathrm{K}}\right],\tag{3.6}$$

where the coefficient c_F of the fermionic contribution remains to be determined. The result for c_F in the $k = 1$ gauge appears to be $c_F = -4$ (see below).

Let us now turn to some technical details of the actual computation of the fermionic graphs we have done. Since the fermions are Majorana (we choose them to be real), the vertices in fermionic bilinears in the action should be antisymmetrized, i.e. M_k in (3.1) should stand for $\frac{1}{2}(M_k - M_k^T)$.[39] Then the 2-loop contributions to the 1-PI Euclidean effective action $\Gamma = -[\ln Z]_{1-PI}$ coming from (3.1) are given symbolically by:[40]

FFB: $-i^2 \times i^3 \times \mathrm{Tr}[M_1 K^{-1}(p)M_1 K^{-1}(-q)]Y_1 Y_1 \Delta^{-1}$

FFBB: $i \times i^2 \times \mathrm{Tr}[M_2 K^{-1}]Y_2 \Delta^{-1}$

FFFF: $-4i \times i^2 \times \left(\mathrm{Tr}[M_3 K^{-1}]\mathrm{Tr}[M_4 K^{-1}] - 2\mathrm{Tr}[M_3 K^{-1}(p)M_4 K^{-1}(-q)]\right)$

The total number of fermionic 2-loop Feynman graphs one needs to evaluate is around few hundred. With the help of a `Mathematica`-based computer program we computed the resulting integrands in the fermionic contributions to the 2-loop effective action represented in the form of the double momentum integrals as in (2.26), (2.32). We found that in the $\theta^1 = k\theta^2$ gauge the integrand depends on the gauge parameter k through the combination

$$\xi = (k - k^{-1})^2\tag{3.7}$$

and, unfortunately, this dependence does not cancel automatically. We have re-arranged the integrands so that to extract power divergences (using transformations of the type $\frac{p^2}{p^2+m^2} = 1 - \frac{m^2}{p^2+m^2}$); the latter were then set to zero by switching on dimensional regularization. We also used the expressions for momentum integrals from appendix E.2. As a result, we found that the $\ln^2 \Lambda \sim \frac{1}{\epsilon^2}$ plus $\ln \Lambda \sim \frac{1}{\epsilon}$ UV divergent part in the 2-loop effective action is coming from (cf. (2.21), (2.36))[41]

$$\bar{\Gamma}_{2F} = \frac{4\pi}{\sqrt{\lambda}}X,\tag{3.8}$$

$$X_\infty = \left(-8[1,2] - (4 - 6\xi)[1,4] + (4 - 2\xi)[1,1]\right) + \left(8[1,2] + 4[1,4]\right)$$
$$+ \frac{2}{3}(4 + \xi)(-36 + 10 + 0 + 40)[1,1] = 6\xi[1,4] + \frac{124 + 22\xi}{3}[1,1].\tag{3.9}$$

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[39]The antisymmetrization should apply also to derivatives in M_k (in Y_2 one should symmetrize them).

[40]If the Minkowski space action is $S = \frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}\theta K\theta + \cdots$ then $e^{iS} = \exp[-\frac{1}{2}\Phi(i\Delta^{-1})^{-1}\Phi - \frac{1}{2}\theta(iK^{-1})^{-1}\theta + \cdots]$.

[41]The resulting effective action computed directly in $d = 2$ contains no IR divergences.

Here the three terms are the contributions of the FFB, FFBB and FFFF graphs, respectively, and

$$[a, b] \equiv I(a)I(b) = \mu^{2\epsilon} \int \frac{d^d p \, d^d q}{(2\pi)^d} \frac{1}{(p^2 + a)(q^2 + b)} \,, \tag{3.10}$$

where $I(a)$ was defined in (2.38). Thus $[a, b]$ contains the $\frac{1}{\epsilon^2} + \frac{1}{\epsilon}$ divergences. The four terms in the last FFFF paranthesis represent the contributions of the $(\bar{\theta}\mathcal{D}\theta)^2$ term in (B.7), of the term with Γ^{ab} in (B.7), of the term with $\Gamma^{a'b'}$ in (B.7) and of the last term in (B.8), respectively. We find that the $[1, 2]$ terms in (3.9) cancel, but there is no cancellation of the remaining terms, contradicting the expected conformal invariance of the theory.

In general, one may expect that in a (globally) supersymmetric theory the regularization of the fermionic and bosonic parts of the action should be done in some consistent way. For bosons we used dimensional regularization, and the cancellation of $\frac{1}{\epsilon^2}$ pole in (2.36), (2.37) ensured also that the remaining $\frac{1}{\epsilon}$ pole had rational coefficient. Even if we would manage to cancel the $\frac{1}{\epsilon^2}$ pole in the fermionic contribution we would then need some sort of dimensional regularization producing d-dependent coefficients so that $\frac{1}{\epsilon}$ pole had rational coefficient to be able to cancel its bosonic counterpart. Which kind of regularization is to be used to ensure that is unclear at the moment. The required rationality of the coefficient of the $\frac{1}{\epsilon}$ pole suggests that the coefficients of $[1, 4]$ and $[1, 1]$ terms in (3.9) should, like coefficient of the $[1, 2]$ term, be separately equal to zero.

Extracting the non-trivial finite part with 3 propagators contained in the FFB contribution we find that it is given by the integral (3.4) (the integral (3.5) does not appear) but its coefficient is also gauge (ξ) dependent

$$X_{\text{fin}} = (4 + 2\xi)I(\sqrt{2}, 1) \tag{3.11}$$

This gauge dependence of the UV divergences and of the finite part which should not be present in the on-shell effective action is indicating a problem with maintaining κ-symmetry at the quantum level in the computational prescription we have used.

Given the unsatisfactory result we found in the $\theta^1 = k\theta^2$ gauge we decided to redo the computation in a light-cone κ-symmetry gauge which is the direct analog of the usual $\Gamma_+ \theta^I = 0$ gauge in which the flat-space GS action becomes quadratic. The quadratic and quartic fermionic terms in the $AdS_5 \times S^5$ action in this gauge are listed in appendix D. Using a similar computational prescription as described above we have obtained the following counterparts of eqs. (3.9) and (3.11)

$$X_\infty = \Big(8[1, 2] + 12[1, 4] + 8[1, 1] \Big) + \Big(-8[1, 2] - 12[1, 4] \Big)$$
$$-\frac{4}{3}(36 + 11 + 24)[1, 1] = -\frac{260}{3}[1, 1] \,, \tag{3.12}$$
$$X_{\text{fin}} = 4I(\sqrt{2}, 1) \,. \tag{3.13}$$

Here the three structures in X_∞ are again the contributions of the FFB, FFBB and FFFF terms. The three terms in the last paranthesis represent the contributions of the (D.16) term, of the first term in (A.4) in the $\mathcal{M}^2$ term in (A.9) and of the second and third terms in (A.4) in (A.9), respectively.

Again, the divergences do not appear to cancel[42] but one piece of good news is that the total coefficients not only of the $[1, 2]$ but also of the $[1, 4]$ structures vanish just as they did in the $k = 1$ ($\xi = 0$) gauge in (3.9). Moreover, the finite term in (3.13) is exactly the same as (3.11) in the $k = 1$ gauge.

If we make the bold assumption that our computational procedure can be corrected so that the results in the two gauges fully agree with all divergences cancelling out and the finite part still given by (3.13) then that would result in the prediction for the coefficient a_2 in (3.6) with $c_F = -4$, i.e.

$$a_2 = a_{2B} + a_{2F} = -\frac{1}{\pi}\left[(1 - 4)\mathrm{K} + \frac{2}{3}\tilde{\mathrm{K}}\right] \approx 0.750335 \ . \tag{3.14}$$

This is even further away from the numerical result (1.3) of [32] than the purely bosonic contribution (2.53).

4. Concluding remarks

In this paper we initiated the study of 2-loop quantum corrections in $AdS_5 \times S^5$ string theory on a particular example of the expansion near a simple "homogeneous" classical string solution. We used conformal gauge for the 2d diffeomorphisms and considered two different choices ("covariant" and "light-cone" ones) for the κ-symmetry gauge.

While we did not manage to completely sort out the expected cancellation of 2-loop UV divergences between the bosonic and the fermionic contributions, our computation revealed the special transcendental structure of the finite term in the 2-loop effective action that determines the next-to-next-to-leading order coefficient a_2 in the strong-coupling expansion of the cusp anomalous dimension on the gauge theory side of the AdS/CFT correspondence. We expect that an improved version of our computation[43] that will resolve the technical problems of apparent gauge dependence and non-cancellation of part of the divergences will not substantially change our conclusion about the finite part determining the structure of the coefficient a_2 in (1.2).

The reason why we have more confidence in our result for the finite rather than divergent part of the 2-loop contribution is that, as explained in section 3, the former is determined only by the quadratic fermionic terms in the $AdS_5 \times S^5$ superstring action (A.8), while the latter depends essentially also on the complicated quartic fermionic terms (A.9).[44]

Our prediction (3.6), (3.14) for the 2-loop coefficient a_2 suggests the following observation. It is interesing to note that the first three terms in the strong coupling expansion

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[42]Power-like divergences have been eliminated in both equations (3.12) and (3.9) due to our regularization scheme. It is, however, interesting to note that in a cutoff-based regularization scheme the power-like divergences appearing in the light-cone gauge are milder than those in the $\theta^1 = k\theta^2$ gauge. In particular, quartic divergences appear to be absent in the former gauge.

[43]One may try to redo the same computation using a different fermionic parametrization of the $AdS_5 \times S^5$ action (e.g., like the one employed in [42]). It would be interesting also to attempt to do a similar computation by starting with the Berkovits formulation [26] of the $AdS_5 \times S^5$ action.

[44]There is of course an issue of apparent gauge dependence of the finite part (3.11) in the $\theta^1 = k\theta^2$ gauge, but given that we got the same finite results in the two very different gauges — $\theta^1 = \theta^2$ and the light-cone gauge — we are inclined to speculate that there is some problem with the computation in the $k \neq 1$ gauge.

of the cusp anomalous dimension (1.2) hint at a systematic expansion in polygamma functions. Indeed, a_1 in (1.2) can be written as $a_1 = -\frac{3}{2\pi}(\Psi(1) - \Psi(\frac{1}{2}))$ and a_2 is a linear combination of K (2.47) and $\tilde{K}$ (2.50) which contain only the values of the first derivative of the digamma function $\Psi(z)$.[45] It is therefore tempting to conjecture that the coefficient a_{n+1} appearing at order $\lambda^{-n/2}$ in the strong coupling expansion in (1.1), (1.2) will be a combination of values of derivatives $\Psi^{(n)}(z)$ at rational arguments. A potentially related structure may follow from the strong coupling expansion of the BFKL kernel which at weak coupling expresses the finite spin twist-2 anomalous dimensions as an expansion in derivatives of the digamma function (see [55] for a comparison between this approach and the Bethe ansatz predictions).

Similar 2-loop computations can also be done for some other special string solutions, for example, for the 2-spin (J_1, J_2) solution in S^5. This solution further simplifies in the limit $J_1 \gg J_2$, and the 1-loop correction vanishes [56]; the same is expected [57] to happen at the two (and higher) loop level. The methods of the present paper allow one to verify this.

Acknowledgments

A. $AdS_5 \times S^5$ superstring Lagrangian

The starting point of the 2-loop computations in this paper is the type IIB Green-Schwarz $AdS_5 \times S^5$ superstring action $I = \int d^2\sigma\, \mathcal{L}$ which is the sum of the "kinetic" and "Wess-Zumino" term [4]

$$\mathcal{L} = \mathcal{L}_{\text{Kin}} + \mathcal{L}_{WZ} = \frac{\sqrt{\lambda}}{2\pi}\left[-\frac{1}{2}\sqrt{-h}h^{\alpha\beta}L_\alpha^A L_\beta^A - 2i\epsilon^{\alpha\beta}\int_0^1 ds\, L_{\alpha s}^A\, \mathrm{s}^{IJ}\bar{\theta}^I\Gamma^A L_{\beta s}^J \right]. \quad \text{(A.1)}$$

The explicit form of this action to quartic order in θ (which is sufficient for our present purpose) was presented in [4]. The exact solution of the Maurer-Cartan equations for the supervielbeine was given in [5] (see also [6]). The $AdS_5 \times S^5$ supersymmetry algebra and thus the resulting string action of [4] can be rewritten in terms of 10d Dirac matrices

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[45]One may wonder if the actual mechanism of cancellation of UV divergences may leave behind a finite piece containing $\ln 2$ terms. The presence of such $\ln 2$ terms could be in conflict with the "transcendentality principle" assuming one extends it from weak-coupling [33, 1] to a strong-coupling expansion. We thank M. Staudacher for this remark.

making it independent of a choice of a particular representation of Γ^A [58] (see also [9, 10] and [39]).[46]

In the above expression $I, J = 1, 2$, $s^{IJ} = (1, -1)$, $L_\alpha^A = (L_{\alpha s}^A)_{s=1}$ and

$$L_{\alpha s}^A = \partial_\alpha x^\rho e_\rho^A(x) - 4i\bar{\theta}^I \Gamma^A \left[\frac{\sinh^2(\frac{s}{2}\mathcal{M})}{\mathcal{M}^2}\right]_{IJ} D_\alpha \theta^J, \qquad L_{\beta s}^J = \left[\frac{\sinh(s\mathcal{M})}{\mathcal{M}} D_\beta \theta\right]^J, \quad \text{(A.2)}$$

$$D\theta^I = \mathcal{D}\theta^I - \frac{i}{2}\epsilon^{IJ}e^A\Gamma_*\Gamma_A\theta^J, \quad \mathcal{D}\theta^I = d\theta^I + \frac{1}{4}\omega^{AB}\Gamma_{AB}\theta^I, \quad e^A = dx^\mu e_\mu^A(x) \quad \text{(A.3)}$$

$$(\mathcal{M}^2)^{IL} = -\epsilon^{IJ}\Gamma_*\Gamma^A\theta^J\bar{\theta}^L\Gamma_A + \frac{1}{2}\epsilon^{LK}(\Gamma^{ab}\theta^I\bar{\theta}^K\Gamma_{ab}\Gamma_* - \Gamma^{a'b'}\theta^I\bar{\theta}^K\Gamma_{a'b'}\Gamma_*') . \quad \text{(A.4)}$$

Here $D^{IJ}D^{JK}\theta^K = 0$. The indices run as follows

$$\mu, \nu = 0, 1, 2, \ldots, 9; \quad A = (a; a') ; \qquad a, b = 0, 1, 2, 3, 4 \; ; \quad a', b' = 5, 6, 7, 8, 9$$

For Dirac matrices we used the notation from [39]

$$\Gamma_* = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4 \qquad \Gamma_*' = i\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_9 \qquad \Gamma_*\Gamma_*' = -\Gamma_*'\Gamma_* = \Gamma_{11}, \qquad \text{(A.5)}$$

$$\Gamma_*^2 = -\Gamma_*'^2 = 1, \qquad \Gamma_{11} = -\Gamma_{0123456789}, \qquad \Gamma_{11}^2 = 1 . \qquad \text{(A.6)}$$

Here Γ_A are 32×32 Dirac matrices, $\Gamma_{(A}\Gamma_{B)} = \eta_{AB} = (-1, +1, \ldots, +1)$, and Γ_{11} defines the 10d chiral projectors. We also assume the standard hermitian conjugation rule for fermions: $(\psi\chi)^\dagger = \chi^\dagger\psi^\dagger$.

In the type IIB string action the fermions are Majorana-Weyl of the same chirality, e.g., $\theta^I = \Gamma_{11}\theta^I$. The Majorana condition

$$\bar{\theta} = \theta^T\mathcal{C}, \qquad \bar{\theta} \equiv \theta^\dagger\Gamma^0, \qquad \mathcal{C}^T = -\mathcal{C} , \qquad \Gamma_A = -\mathcal{C}^{-1}\Gamma_A^T\mathcal{C} . \qquad \text{(A.7)}$$

can be solved by choosing $\mathcal{C} = \Gamma^0$ and thus having θ real.[47] In the specific representation of Γ-matrices used in [4, 39] $\Gamma_{11} = I_{16} \times \sigma_3$, so that "left" spinors satisfying $\theta^I = \Gamma_{11}\theta^I$ have lower 16 components equal to zero. The final result of our computation should not depend on a choice of a particular representation of Γ_A and $\mathcal{C}$.

To quartic order in fermions the fermionic part of (A.1) is ($\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$, $\mathcal{L}_F = \mathcal{L}_{F2} + \mathcal{L}_{F4} + \cdots$)

$$\frac{2\pi}{\sqrt{\lambda}}\mathcal{L}_{F2} = i(\eta^{\alpha\beta}\delta^{IJ} - \epsilon^{\alpha\beta}s^{IJ})\bar{\theta}^I\slashed{e}_\alpha D_\beta\theta^J$$

$$= i(\eta^{\alpha\beta}\delta^{IJ} - \epsilon^{\alpha\beta}s^{IJ})\bar{\theta}^I\slashed{e}_\alpha\left[\delta^{JK}\mathcal{D}_\beta - \frac{i}{2}\epsilon^{JK}\Gamma_*\slashed{e}_\beta\right]\theta^K , \qquad \text{(A.8)}$$

$$\frac{2\pi}{\sqrt{\lambda}}\mathcal{L}_{F4} = (\eta^{\alpha\beta}\delta^{IJ} - \epsilon^{\alpha\beta}s^{IJ})\left[\frac{i}{12}\bar{\theta}^I\slashed{e}_\alpha\mathcal{M}_{JK}^2 D_\beta\theta^K + \frac{1}{2}(\bar{\theta}^K\Gamma^A D_\alpha\theta^K)(\bar{\theta}^I\Gamma_A D_\beta\theta^J)\right] \quad \text{(A.9)}$$

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[46]This "10d covariant" form of the action naturally comes out of the general form of GS action in type IIB supergravity background [3] once one specifies the curvature and the 5-form field to their $AdS_5 \times S^5$ values.

[47]For 10d Majorana fermions of the same chirality $\bar{\psi}_1\Gamma_{A_1\ldots A_n}\psi_2$ is non-zero for n=odd and is symmetric in ψ_1, ψ_2 for $n = 3, 7$ and antisymmetric if $n = 1, 5, 9$.

Here we used the conformal gauge $\sqrt{-h}h^{\alpha\beta} = \eta^{\alpha\beta}$ and

$$\not{e}_\alpha = e_\alpha^A \Gamma_A \,, \qquad e_\alpha^A = e_\rho^A \partial_\alpha x^\rho \,, \qquad \mathcal{D}_\beta = \partial_\beta + \frac{1}{4}\omega_\beta{}^{AB}\Gamma_{AB} \,, \qquad \omega_\alpha{}^{AB} = \omega_\rho{}^{AB}\partial_\alpha x^\rho \,. \quad \text{(A.10)}$$

The metric, vielbeine and spin connection are those following from the $AdS_5 \times S^5$ metric (2.12). In particular, the non-zero background values are (see (2.12), (2.16), (A.20))[48]

$$\not{e}_0 = \frac{\kappa}{\sqrt{2}}(\Gamma_8 + \Gamma_9) \,, \qquad \not{e}_1 = -\frac{i\kappa}{\sqrt{2}}(\Gamma_8 - \Gamma_9) \,, \qquad e_\alpha^A e_{A\beta} = -\kappa^2 \eta_{\alpha\beta} \,, \quad \text{(A.11)}$$

$$\omega_0{}^{AB}\Gamma_{AB} = \sqrt{2}\kappa\Gamma_7(\Gamma_8 - \Gamma_9) = 2i\Gamma_7\not{e}_1 \,, \qquad \omega_1{}^{AB}\Gamma_{AB} = -i\sqrt{2}\kappa\Gamma_7(\Gamma_8 + \Gamma_9) = -2i\Gamma_7\not{e}_0 \,.$$

Let us list the general expressions for the projected vielbeine $e_\alpha^A = e_\rho^A \partial_\alpha x^\rho$ for the $AdS_5 \times S^5$ metric in (2.13), (2.14)

$$e_\alpha^0 = \frac{1 + \frac{1}{4}z^2}{1 - \frac{1}{4}z^2}\partial_\alpha t \,, \qquad\qquad e_\alpha^k = \frac{1}{1 - \frac{1}{4}z^2}\partial_\alpha z^k \,, \qquad\qquad k = 1,2,3,4 \quad \text{(A.12)}$$

$$e_\alpha^5 = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2-y^2}}\partial_\alpha x + \frac{xy}{\sqrt{(1-y^2)(1-x^2-y^2)}}\partial_\alpha y \,, \qquad e_\alpha^6 = \frac{\partial_\alpha y}{\sqrt{1-y^2}} \quad \text{(A.13)}$$

$$e_\alpha^7 = \sqrt{1-x^2-y^2}\partial_\alpha\psi \,, \quad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(A.14)}$$

$$e_\alpha^8 = \sqrt{1-x^2-y^2}\cos\psi\partial_\alpha\phi_2 \,, \qquad e_\alpha^9 = \sqrt{1-x^2-y^2}\sin\psi\partial_\alpha\phi_3 \,. \quad \text{(A.15)}$$

The Lorentz connection satisfying $\epsilon^{\alpha\beta}(\partial_\alpha e_\beta^A + \omega_\alpha{}^A{}_B e_\beta^B) = 0 \,(\omega_\alpha^{AB} \equiv \omega_\rho{}^{AB}\partial_\alpha x^\rho = -\omega_\alpha^{AB})$ is

$$\omega_\alpha{}^{0i} = \partial_\alpha t \frac{z^k}{1 - \frac{1}{4}z^2} \,, \qquad \omega_\alpha{}^{kn} = -\frac{1}{2}\frac{z^k\,\partial_\alpha z^n - z^n\,\partial_\alpha z^k}{1 - \frac{1}{4}z^2} \,, \qquad \omega_\alpha{}^{56} = -\frac{y}{\sqrt{1-y^2}}e_\alpha^5 \quad \text{(A.16)}$$

$$\omega_\alpha{}^{57} = \frac{x\partial_\alpha\psi}{\sqrt{1-y^2}} \,, \qquad\qquad \omega_\alpha{}^{58} = \frac{x\cos\psi\,\partial_\alpha\phi_2}{\sqrt{1-y^2}} \qquad\qquad\qquad\qquad \text{(A.17)}$$

$$\omega_\alpha{}^{59} = \frac{x\sin\psi\,\partial_\alpha\phi_3}{\sqrt{1-y^2}} \,, \qquad\qquad \omega_\alpha{}^{67} = \frac{y\sqrt{1-x^2-y^2}\partial_\alpha\psi}{\sqrt{1-y^2}} \qquad\qquad\qquad \text{(A.18)}$$

$$\omega_\alpha{}^{68} = \frac{y\sqrt{1-x^2-y^2}\cos\psi\,\partial_\alpha\phi_2}{\sqrt{1-y^2}} \,,$$

$$\omega_\alpha{}^{69} = \frac{y\sqrt{1-x^2-y^2}\sin\psi\,\partial_\alpha\phi_3}{\sqrt{1-y^2}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(A.19)}$$

$$\omega_\alpha{}^{78} = \sin\psi\,\partial_\alpha\phi_2 \,, \qquad\qquad \omega_\alpha{}^{79} = -\cos\psi\,\partial_\alpha\phi_3 \,. \qquad\qquad\qquad\qquad \text{(A.20)}$$

B. κ-symmetry gauge fixing: $\theta^1 = k\theta^2$ gauge

One natural gauge choice (used also in [9, 10, 12]) in the present case is [49]

$$\theta^1 = \theta^2 \equiv \theta \,. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.1)}$$

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[48]We recall that the AdS_5 and S^5 coordinates in (2.13), (2.14) are labeled as $0,1,2,3,4$ and $5,6,7,8,9$.

[49]This gauge is singular if one expands near a null geodesic but is regular if the string background has both τ and σ dependence.

Then for the relevant (to 2-loop order) quartic terms in the fermions one finds

$$L_{\alpha s}^{A} = \partial_\alpha x^\rho e_\rho^A - 2is^2\bar\theta\Gamma^A \mathcal{D}_\alpha\theta + \frac{s^4}{12}\bar\theta\Gamma^A(-\Gamma^{ab}\theta\bar\theta\Gamma_{ab}\Gamma_* + \Gamma^{a'b'}\theta\bar\theta\Gamma_{a'b'}\Gamma_*')\Gamma_*\Gamma_B\theta \; e_\alpha^B$$

$$= \partial_\alpha x^\rho e_\rho^A - 2is^2\bar\theta\Gamma^A \mathcal{D}_\alpha\theta + \frac{s^4}{12}\bar\theta\Gamma^A(-\Gamma^{ab}\theta\bar\theta\Gamma_{ab} + \Gamma^{a'b'}\theta\bar\theta\Gamma_{a'b'})\Gamma_B\theta \; e_\alpha^B$$

$$\mathrm{s}^{IJ}\bar\theta^I\Gamma^A L_{\beta s}^{J} = -is\bar\theta\Gamma^A\Gamma_*\Gamma_B\theta \; e_\beta^B - \frac{2s^3}{3}\bar\theta\Gamma^A\Gamma_*\Gamma^B\theta\bar\theta\Gamma_B \mathcal{D}_\beta\theta \; . \tag{B.2}$$

As a result, the "kinetic" and "WZW" parts of (A.1) become (to order θ^4)

$$\frac{2\pi}{\sqrt\lambda}\mathcal{L}_{\mathrm{Kin}} = \eta^{\alpha\beta}\left[-\frac{1}{2}\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}(x) + 2ie_\alpha^A\bar\theta\Gamma_A \mathcal{D}_\beta\theta + 2\bar\theta\Gamma^A \mathcal{D}_\alpha\theta\bar\theta\Gamma_A \mathcal{D}_\beta\theta \right.$$
$$\left. +\frac{1}{12}e_\alpha^A e_\beta^B \; \bar\theta\Gamma_A(\Gamma^{ab}\theta\bar\theta\Gamma_{ab} - \Gamma^{a'b'}\theta\bar\theta\Gamma_{a'b'})\Gamma_B\theta \right], \tag{B.3}$$

$$\frac{2\pi}{\sqrt\lambda}\mathcal{L}_{\mathrm{WZ}} = \epsilon^{\alpha\beta}\left[-e_\alpha^A e_\beta^B\bar\theta\Gamma_A\Gamma_*\Gamma_B\theta + \frac{i}{3}e_\alpha^A\bar\theta\Gamma_A\Gamma_*\Gamma_B\theta\bar\theta\Gamma^B \mathcal{D}_\beta\theta - ie_\alpha^A\bar\theta\Gamma_B\Gamma_*\Gamma_A\theta\bar\theta\Gamma^B \mathcal{D}_\beta\theta \right]$$

$$= \epsilon^{\alpha\beta}\left[-e_\alpha^A e_\beta^B\bar\theta\Gamma_A\Gamma_*\Gamma_B\theta + \frac{4i}{3}e_\alpha^A\bar\theta\Gamma_A\Gamma_*\Gamma_B\theta\bar\theta\Gamma^B \mathcal{D}_\beta\theta \right]. \tag{B.4}$$

We used that for the "left" fermions $\Gamma_{11}\theta = \Gamma_*\Gamma_*'\theta = \theta$ and also that $\bar\theta\Gamma_B\Gamma_*\Gamma_A\theta = -\bar\theta\Gamma_A\Gamma_*\Gamma_B\theta$. The resulting action is the same as the quartic fermionic action found in eqs. (4.12)-(4.14) in [4] upon restricting it to the gauge (B.1).

One may also consider a more general gauge (here k is a real number)

$$\theta^1 = k\,\theta^2, \qquad \theta^2 \equiv \theta\,. \tag{B.5}$$

Then to θ^4 order

$$L_{\alpha s}^{A} = \partial_\alpha x^\rho e_\rho^A - i(1+k^2)s^2\bar\theta\Gamma^A \mathcal{D}_\alpha\theta$$
$$+(1+k^2)^2\frac{s^4}{48}\bar\theta\Gamma^A(-\Gamma^{ab}\theta\bar\theta\Gamma_{ab} + \Gamma^{a'b'}\theta\bar\theta\Gamma_{a'b'})\Gamma_C\theta e_\alpha^C\,,$$

$$\mathrm{s}^{IJ}\bar\theta^I\Gamma^A L_{\beta s}^{J} = (k^2-1)s\bar\theta\Gamma^A \mathcal{D}_\beta\theta - iks\bar\theta\Gamma^A\Gamma_*\Gamma_B\theta e_\beta^B - k(1+k^2)\frac{s^3}{3}\bar\theta\Gamma^A\Gamma_*\Gamma^B\theta\bar\theta\Gamma_B \mathcal{D}_\beta\theta$$
$$+(k^4-1)\frac{is^3}{24}\bar\theta\Gamma_A(-\Gamma^{ab}\theta\bar\theta\Gamma_{ab} + \Gamma^{a'b'}\theta\bar\theta\Gamma_{a'b'})\Gamma_C\theta e_\beta^C \; . \tag{B.6}$$

As a result, (B.3) and (B.4) are generalized to

$$\frac{2\pi}{\sqrt\lambda}\mathcal{L}_{\mathrm{Kin}} = \eta^{\alpha\beta}\left[-\frac{1}{2}\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}(x) + i(1+k^2)e_\alpha^A\bar\theta\Gamma^A \mathcal{D}_\beta\theta + \frac{(1+k^2)^2}{2}\bar\theta\Gamma^A \mathcal{D}_\alpha\theta\bar\theta\Gamma^A \mathcal{D}_\beta\theta \right.$$
$$\left. -\frac{(1+k^2)^2}{48}e_\alpha^A e_\beta^B\bar\theta\Gamma_A(-\Gamma^{ab}\theta\bar\theta\Gamma_{ab} + \Gamma^{a'b'}\theta\bar\theta\Gamma_{a'b'})\Gamma_B\theta \right], \tag{B.7}$$

$$\frac{2\pi}{\sqrt\lambda}\mathcal{L}_{\mathrm{WZ}} = \epsilon^{\alpha\beta}\left[-i(k^2-1)e_\alpha^A\bar\theta\Gamma_A \mathcal{D}_\beta\theta - ke_\alpha^A e_\beta^B\bar\theta\Gamma_A\Gamma_*\Gamma_B\theta \right.$$
$$\left. +\frac{1}{6}ik(k^2+1)e_\alpha^A\bar\theta\Gamma_A\Gamma_*\Gamma_B\theta\bar\theta\Gamma^B \mathcal{D}_\beta\theta - \frac{1}{2}ik(k^2+1)e_\alpha^A\bar\theta\Gamma_B\Gamma_*\Gamma_A\theta\bar\theta\Gamma^B \mathcal{D}_\beta\theta \right]$$

$$= \epsilon^{\alpha\beta}\left[-i(k^2-1)e_\alpha^A\bar\theta\Gamma_A \mathcal{D}_\beta\theta - ke_\alpha^A e_\beta^B\bar\theta\Gamma_A\Gamma_*\Gamma_B\theta \right.$$
$$\left. +\frac{2}{3}ik(k^2+1)e_\alpha^A\bar\theta\Gamma_A\Gamma_*\Gamma_B\theta\bar\theta\Gamma^B \mathcal{D}_\beta\theta \right], \tag{B.8}$$

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where we used that the term proportional to $k^4 - 1$ vanishes under antisymmetrization in α, β. Note that if we rescale θ by $(\frac{k^2+1}{2})^{1/2}$ then (B.7) will become equivalent to (B.3) while (B.8) will take the form

$$\frac{2\pi}{\sqrt{\lambda}}\mathcal{L}_{\mathrm{WZ}} = \epsilon^{\alpha\beta}\left[-2i\frac{k^2-1}{k^2+1}e_\alpha^A\bar{\theta}\Gamma_A\mathcal{D}_\beta\theta - \frac{2k}{k^2+1}e_\alpha^A e_\beta^B\bar{\theta}\Gamma_A\Gamma_*\Gamma_B\theta \right.$$
$$\left. + \frac{8i}{3}\frac{k}{k^2+1}e_\alpha^A\bar{\theta}\Gamma_A\Gamma_*\Gamma_B\theta\bar{\theta}\Gamma^B\mathcal{D}_\beta\theta \right], \tag{B.9}$$

which reduces to (B.4) for $k = 1$.

The fermionic propagator in the $\theta^1 = k\theta^2$ gauge corresponding to (B.3), (B.9) (after the above rescaling of θ and after the rescaling of momenta by κ, i.e. with the same normalization as in (3.3)) is

$$K^{-1}(q) = \frac{k^{-1}+k}{8\sqrt{2}\,(q^2+1)}\left(\left[\frac{k^{-1}}{2}(1-i)(q_0-q_1) + \frac{k}{2}(1+i)(q_0+q_1)\right]\Gamma_8 \right.$$
$$\left. + \left[\frac{k^{-1}}{2}(1+i)(q_0-q_1) + \frac{k}{2}(1-i)(q_0+q_1)\right]\Gamma_9 - i\sqrt{2}\,\Gamma_*\Gamma_8\Gamma_9\right)\mathcal{C} \quad \text{(B.10)}$$

where q is the 2d momentum and $\mathcal{C}$ is the charge conjugation matrix. Note that the contribution of the connection terms in $\mathcal{D}_\alpha$ to the propagator vanishes (cf. (A.11)).

The propagator is invariant under $k \to k^{-1}$ combined with the 2d parity transformation, i.e. $q_1 \to -q_1$. The same transformation is also a symmetry of the interaction terms in (B.7), (B.8).[50]

C. Cancellation of 2-loop corrections in flat-space Green-Schwarz action in $\theta^1 = \theta^2$ gauge

To clarify the issue of cancellation of power divergences in diagrams with fermion lines it is useful to consider a similar 2-loop cancellation in flat space type IIB GS action [2] (cf. (A.1))

$$I = \frac{1}{2\pi\alpha'}\int d^2\sigma\left[-\frac{1}{2}(\partial_\alpha x^\mu - i\bar{\theta}^I\Gamma^\mu\partial_\alpha\theta^I)^2 - i\epsilon^{\alpha\beta}\mathrm{s}^{IJ}\bar{\theta}^I\Gamma_\mu\partial_\beta\theta^J\left(\partial_\alpha x^\mu - \frac{1}{2}i\bar{\theta}^K\Gamma^\mu\partial_\alpha\theta^K\right)\right], \text{(C.1)}$$

where we fixed the conformal gauge $\sqrt{-h}h^{\alpha\beta} = \eta^{\alpha\beta}$. Let us expand this action near the "homogeneous" classical solution[51]

$$x^\mu = N_\alpha^\mu\sigma^\alpha, \qquad \sigma^\alpha = (\tau,\sigma), \tag{C.2}$$

---

[50]Note that the GS action (A.1) is not invariant under $\theta^1 \to \theta^2$ due to: (i) the presence of s^{IJ} in the WZ term, and (ii) the presence of ϵ^{IJ} terms in $D\theta$ and in $\mathcal{M}$ in (A.4). The first reason is present already in flat-space GS action and can be compensated by 2d parity transformation or $\epsilon^{\alpha\beta} \to -\epsilon^{\alpha\beta}$. The second is due to the presence of a non-trivial RR background: each ϵ^{IJ} factor is accompanied by a factor of Γ_* (note that $\Gamma'_* = \Gamma_*\Gamma_{11}$) which is present due to coupling to self-dual F_5 field. Thus reversing the sign of F_5 background corresponds to $\theta^1 \to \theta^2$ combined with 2d parity transformation.

[51]Since the above action depends on x^μ only through its derivatives, the coefficients in the expanded action will be constant.

where N_α^μ are constant vectors (which we may formally allow to be complex) assumed to satisfy

$$\partial_\alpha x^\mu \partial_\beta x_\mu = \eta_{\mu\nu} N_\alpha^\mu N_\beta^\nu = \mathrm{f}\eta_{\alpha\beta} \ . \tag{C.3}$$

Here f is a background-dependent constant. The direct analog of our S^5 background in (2.16) is the following choice

$$N_\alpha^2 = \frac{\kappa}{\sqrt{2}}(1, -i) \,, \qquad N_\alpha^3 = \frac{\kappa}{\sqrt{2}}(1, i) \,, \qquad \mathrm{f} = -\kappa^2 \,, \tag{C.4}$$

where x^2, x^3 directions are analogs of ϕ_2 and ϕ_3 in (2.16).

Let us fix the κ-symmetry by the same condition as in (B.1): $\theta^1 = \theta^2 \equiv \theta$. Since $\mathrm{s}^{IJ} = (1, -1)$, the contribution of the WZ term in (C.1) then vanishes. The resulting fermionic kinetic term will turn out to be non-degenerate so this gauge is admissible.

Setting $x^\mu \to x^\mu + \tilde{x}^\mu$, we get the following action for the fluctuations $\tilde{x}^\mu$, θ

$$\tilde{I} = \frac{1}{2\pi\alpha'} \int d^2\sigma \left[-\frac{1}{2}(\partial_\alpha \tilde{x}^\mu - 2i\bar\theta\Gamma^\mu\partial_\alpha\theta)^2 + 2i\bar\theta\gamma^\alpha\partial_\alpha\theta \right] \,, \tag{C.5}$$

where

$$\gamma_\alpha \equiv N_\alpha^\mu\Gamma_\mu \,, \qquad \Gamma_{(\mu}\Gamma_{\nu)} = \eta_{\mu\nu} \,, \qquad \gamma_{(\alpha}\gamma_{\beta)} = \mathrm{f}\eta_{\alpha\beta} \ . \tag{C.6}$$

To this action we should add the contribution of the conformal gauge ghosts and the κ-symmetry ghosts. The former is decoupled from the background but the latter is non-trivial. The invariance of the GS action under the κ-symmetry $\delta\theta^I = (\partial_\alpha x^\mu - i\bar\theta^J\Gamma^\mu\partial_\alpha\theta^J)\Gamma_\mu\kappa^{\alpha I}$ (here the spinor parameter $\kappa^{\alpha 1}$ is selfdual and $\kappa^{\alpha 2}$ — antiselfdual in 2d vector index α) leads in the $\theta^1 = \theta^2$ gauge to an ultralocal ghost action[52]

$$I_{gh}(b, c) = \frac{1}{2\pi\alpha'} \int d^2\sigma \ b^I (N_\alpha^\mu + \partial_\alpha \tilde{x}^\mu - 2i\bar\theta\Gamma^\mu\partial_\alpha\theta)\Gamma_\mu \ c^{\alpha I} \ . \tag{C.7}$$

On general grounds, one should expect that the total string partition function should be trivial despite the non-linearity of the action (C.5). Indeed, we could have fixed first the conformal gauge $x^+ = p^+\tau$, $\Gamma_+\theta^I = 0$ in which the GS action (C.1) becomes quadratic and then choose the background (C.2) in the x^2, x^3 directions transverse to (x^+, x^-), $x^\pm = x^0 \pm x^1$. Since we are expanding near an on-shell background, the partition function should be gauge-independent, i.e. still trivial.

Let us note that the resulting theory (C.5) is formally non-renormalizable: the fermionic kinetic term is linear in 2d momentum while fermionic interactions contain derivatives. This is a reflection of the absence of the (non-unitary) $\partial\bar\theta\partial\theta$ kinetic term in the GS action (i.e. of the degeneracy of the corresponding superspace sigma model metric). Thus we should expect divergences with higher powers of the UV cutoff (in an appropriate covariant regularization); the triviality of quantum corrections requires cancellation of all divergences, and, in particular, the absence of logarithmic divergences.

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[52]The conformal gauge ghosts and the κ-symmetry ghosts decouple.

Let us first consider the 1-loop approximation. Counting non-trivial $\frac{1}{2}\ln\det(-\partial^2)$ contributions one gets 10 from bosons, -2 from conformal ghosts and $-\frac{1}{2}\times 16 = 8$ from one MW fermion θ; this checks that the total effective number of degrees of freedom is 0. In addition, there is a quadratic divergence proportional to $\ln f$ coming from the θ-determinant $((\gamma^\alpha\partial_\alpha)^2 = f\partial^2)$. It is cancelled by the 1-loop contribution of the κ-ghosts in (C.7) $(\int d^2\sigma\ b^I\gamma_\alpha c^{I\alpha} + \cdots)$.[53]

To compute the 2-loop contribution it is useful first to transform the action (C.5), (C.7) into an equivalent but simpler-looking 2-d dual (or "T-dual") form[54] by introducing two auxiliary fields L_α^μ and P_μ^α and writing the total fluctuation action as

$$I_{\rm tot} = \frac{1}{2\pi\alpha'}\int d^2\sigma\Big[-\frac{1}{2}(L_\alpha^\mu)^2 + 2i\bar\theta\gamma^\alpha\partial_\alpha\theta$$
$$+ b^I(\gamma_\alpha + L_\alpha^\mu\Gamma_\mu)c^{I\alpha} + P_\mu^\alpha[L_\alpha^\mu - (\partial_\alpha\tilde x^\mu - 2i\bar\theta\Gamma^\mu\partial_\alpha\theta)]\Big]\ . \quad \text{(C.8)}$$

Integrating first over $\tilde x^\mu$ (implying $P_\mu^\alpha = \epsilon^{\alpha\beta}\partial_\beta y_\mu$ where y^μ is a "2-d dual" of $\tilde x^\mu$) and then over L_α^μ results in

$$\tilde I_{\rm tot} = \frac{1}{2\pi\alpha'}\int d^2\sigma\Big[-\frac{1}{2}(\partial_\alpha y^\mu + \epsilon_{\alpha\beta}b^I\Gamma^\mu c^{I\beta})^2 + 2i\bar\theta\gamma^\alpha\partial_\alpha\theta - 2i\epsilon^{\alpha\beta}\partial_\beta y^\mu\bar\theta\Gamma_\mu\partial_\alpha\theta$$
$$+ b^I\gamma_\alpha c^{I\alpha} + \epsilon_{\alpha\beta}\partial^\beta y^\mu b^I\Gamma_\mu c^{I\alpha}\Big]\ . \quad \text{(C.9)}$$

This can be written also as

$$\tilde I_{\rm tot} = \frac{1}{2\pi\alpha'}\int d^2\sigma\Big[-\frac{1}{2}(\partial_\alpha y^\mu)^2 + 2i\bar\theta\gamma^\alpha\partial_\alpha\theta + b^I\gamma_\alpha c^{I\alpha}$$
$$- 2i\epsilon^{\alpha\beta}\partial_\beta y^\mu\bar\theta\Gamma_\mu\partial_\alpha\theta + \epsilon_{\alpha\beta}\partial^\alpha y^\mu b^I\Gamma_\mu c^{I\beta} + \frac{1}{2}(b^I\Gamma^\mu c^{I\alpha})^2\Big]\ . \text{(C.10)}$$

An advantage of this form of the action is the absence of the θ^4 and $bc\theta^2$ terms at the price of the appearance of (simpler) $(bc)^2$ term.[55]

Then the only 2-loop diagram involving θ is then of type (a) in figure 1 where one line is bosonic and two lines are fermionic. Because of the properties of γ_α in (C.6) the propagator for the Majorana-Weyl 10d spinor θ is essentially the same as for a 2-d fermion, i.e. is (in momentum representation) $\frac{p^\alpha\gamma_\alpha}{p^2}$. Then the non-trivial contribution (from the diagram on figure 1(a)) to the 2-loop effective action is proportional to (V_2 is the 2d volume factor)

$$\frac{V_2}{f^2}\int \frac{d^2p\,d^2q}{(2\pi)^4}\frac{{\rm Tr}(\Gamma^\mu p^\alpha\gamma_\alpha\Gamma_\mu q^\beta\gamma_\beta)\ \epsilon^{\gamma\delta}p_\gamma(p+q)_\delta\epsilon^{\gamma'\delta'}q_{\gamma'}(p+q)_{\delta'}}{p^2q^2(p+q)^2}\ . \quad \text{(C.11)}$$

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[53]Similar cancellation applies to the p^+-dependence in lightcone gauge.

[54]A similar transformation was used in [8].

[55]To make the structure of possible cancellations more transparent it might be useful to replace the (anti)selfdual ghost $c^{I\alpha}$ with two *commuting* ghost spinor fields (the associated Jacobian is background-independent): $c^{1\alpha} = (\eta^{\alpha\beta} + \epsilon^{\alpha\beta})\partial_\beta\vartheta^1$, $c^{2\alpha} = (\eta^{\alpha\beta} - \epsilon^{\alpha\beta})\partial_\beta\vartheta^2$. That way it may be possible to show the cancellation of corrections between loops of θ and loops of (b, ϑ^I) to all orders. We will not pursue this here.

Since[56] $\mathrm{Tr}(\Gamma^\mu p^\alpha \gamma_\alpha \Gamma_\mu q^\beta \gamma_\beta) = -10 \times 16\, \mathrm{f}(pq)$ we end up with (omitting the prefactor $\frac{V_2}{\mathrm{f}}$)

$$\int \frac{d^2p\, d^2q}{(2\pi)^4} \frac{(pq)[(pq)^2 - p^2q^2]}{p^2q^2(p+q)^2} = \int \frac{d^2p\, d^2q}{4(2\pi)^4}\left[1 + 3\frac{p^2}{q^2} - 3\frac{(p+q)^2}{q^2}\right.$$
$$\left. + \frac{q^2}{(p+q)^2} + \frac{(p^2+q^2+2pq)^2}{2p^2q^2} - \frac{1}{2}\frac{p^4}{(p+q)^2q^2}\right] \text{(C.12)}$$

where we factorized the integrand and used the symmetry under $p \to q$ as well as Lorentz invariance of the integrand. The above integral can be simplified further into

$$\int \frac{d^2p\, d^2q}{4(2\pi)^4}\left[\frac{p^2}{q^2} + \frac{q^2}{(p+q)^2} - \frac{1}{2}\frac{p^4}{(p+q)^2q^2}\right] . \tag{C.13}$$

This integral is quartically divergent. Applying the dimensional regularization (in combination with an IR regularization by a mass, see [40]) we conclude that it does not contain any logarithmically divergent or finite parts, i.e. the result vanishes. The contribution of ghosts is also trivial in dimensional regularization.

Alternatively, we may use an explicit regularization like an exponential cutoff by inserting $e^{-\frac{p^2}{\Lambda^2}}$ for each momentum integral. Then we get for (C.12) (omitting the overall factor)

$$\int \frac{d^2p\, d^2q\, d^2k}{(2\pi)^4} \delta^{(2)}(p+q+k)\frac{(k^2-p^2-q^2)[(k^2-p^2-q^2)^2 - 4\,p^2q^2]}{8p^2q^2k^2}e^{-\frac{1}{\Lambda^2}(p^2+q^2+k^2)} \text{ (C.14)}$$

Using the symmetry of the integrand under interchange of p, q, k we obtain

$$\int \frac{d^2p\, d^2q\, d^2k}{4(2\pi)^4}\delta^{(2)}(p+q+k)\left[1 - \frac{k^4}{2p^2q^2} + \frac{k^2}{p^2}\right]e^{-\frac{1}{\Lambda^2}(p^2+q^2+k^2)}$$
$$= \int \frac{d^2p\, d^2q}{4(2\pi)^4}\left[1 - \frac{(p+q)^4}{2p^2q^2} + \frac{(p+q)^2}{p^2}\right]e^{-\frac{1}{\Lambda^2}(p^2+q^2+(p+q)^2)} . \tag{C.15}$$

Evaluating the integrals here we find that the first term in the bracket gives $\frac{\Lambda^4}{192\,\pi^2}$ while each of the last two gives zero.

The result is thus simply a quartic divergence, which should then be cancelled against the local κ-symmetry ghost contribution so that the total 2-loop contribution to the effective action is trivial. A careful check of this cancellation may require a systematic development of the phase-space quantization of the GS action in the $\theta^1 = \theta^2$ gauge (with all measure factors taken into account).[57] The use of dimensional regularization allows one to by-pass this problem. This is the strategy we adopt also in the curved-space case considered in this paper.

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[56]The trace is taken with the Weyl projector implied.

[57]In general, local measure may not be fixed in the Lagrangian quantization; that means also power divergences can not be cancelled unless all local factors of ghosts and measure are included. For a previous discussion of quantization of flat-space GS action see, e.g., [59].

D. κ-symmetry light-cone gauge $\Gamma_+\theta^I = 0$

The flat-space GS action is known to simplify dramatically in the κ-symmetry light-cone gauge $\Gamma_+\theta^I = 0$: the quartic fermionic term in it vanishes. It is natural to expect that a choice of a similar gauge may also lead to important simplifications in curved space-time case. In particular, at least part of power divergences may then be absent. Below we shall present the details of the structure of the $AdS_5 \times S^5$ action in a light-cone gauge $\Gamma_+\theta^I = 0$ needed for computing the fermionic 2-loop contribution discussed in section 3.

D.1 Vanishing of 2-loop correction in the expansion near null geodesic

As a preparation for the 2-loop computation we are interested in it is useful first to consider the expansion near the simplest point-like string configuration: null geodesic that goes around S^5. Since this is a BPS configuration preserving 1/2 of supersymmetry one expects to find that all world-sheet loop contributions to the sigma model partition function expanded near this background vanish, i.e. the ground-state energy should not receive quantum corrections. This is indeed easily verified in the 1-loop approximation where choosing the light-cone κ-symmetry gauge one gets 8 bosonic and 8 fermionic fluctuation modes with equal mass [7, 41, 10]. We have checked explicitly that the same is true also in the 2-loop approximation where one no longer has a benefit of an effective 2d supersymmetry or even manifest 2d Lorentz symmetry present in the "1-loop" (i.e. "plane-wave") action.

We shall use conformal gauge and consider the expansion of the superstring action near the following sigma model solution corresponding to the metric (2.13), (2.14): $t = \kappa\tau$, $\phi_2 = \kappa\tau$ with all other angles being trivial. It is actually useful to change the parametrization of the S^5 metric from (2.14) to the one similar to (2.13):

$$(ds^2)_{S^5} = \left(\frac{1 - \frac{1}{4}y^2}{1 + \frac{1}{4}y^2}\right)^2 d\phi^2 + \frac{dy^n dy_n}{(1 + \frac{1}{4}y^2)^2}, \qquad n = 1, 2, 3, 4 . \tag{D.1}$$

Then the classical solution (which solves both the sigma model equations and the conformal gauge constraints) is

$$t = \kappa\tau, \qquad \phi = \kappa\tau, \qquad z_k = 0, \qquad y_n = 0, \tag{D.2}$$

and we should thus expand the action to quartic order in fluctuation fields $\tilde{t} = t - \kappa\tau$, $\tilde{\phi} = \phi - \kappa\tau$, z_k, y_n and θ^I subject to the l.c. κ-symmetry gauge condition $(\Gamma_0 + \Gamma_5)\theta^I = 0$ (we label ϕ as the 5-th coordinate).

Let us first make general comments on the bosonic contribution. The logarithmically divergent parts of the effective actions of the decoupled AdS_5 and S^5 sigma models are each given by the counterterm (2.7) multiplying the $\partial x \partial x$ term. For a symmetric space (2.7) is proportional to the metric itself, so we get, up to numerical coefficients, $(\alpha' R + \alpha'^2 R^2 + \alpha'^3 R^3 + \cdots)G_{\mu\nu}(x)\partial x^\mu \partial x^\nu$. Since the scalar curvatures of AdS_5 and S^5 here are opposite in sign, we conclude that the divergence at one (or any odd) loop is proportional to the *difference* of the AdS_5 and S^5 classical actions, while the divergence at two (or any

even) loop is proportional to the *sum* of the AdS_5 and S^5 classical actions (i.e. to the total classical string action). The difference of the AdS_5 and S^5 classical actions is non-vanishing on (D.2), in agreement with the presence of 1-loop divergence coming from 8 equal-mass bosonic modes; this divergence is of course cancelled by the fermions. The sum of the AdS_5 and S^5 classical actions vanishes on the solution (D.2), so we conclude that the bosonic part of the partition function can get only finite contribution at two (or any even number of) loops.

This is indeed what we have found by the direct 2-loop computation: the bosonic 2-loop contribution happens to be completely trivial, i.e. the 2-loop bosonic part of the effective action vanishes.[58]

As for the fermionic part, we found (using the l.c. gauge expansion) that the contribution of the diagram in figure 1(a) with two Yukawa FFB vertices is identically zero, while the contributions of the FFBB and FFFF terms in figure 1(b) are proportional to the square of the simple massive tadpole integral[59] $[1,1]$ in (3.10) with the coefficients being, respectively, 32 and -32.

Thus the total 2-loop term in the effective action expanded near the null geodesic is indeed zero.

Let us stress that to arrive at this result we used dimensional regularization only in a limited sense: all tensor algebra was done in $d = 2$ and we continued to $d < 2$ (to eliminate power divergences) only at the very end for the scalar integrals found after factorization of highest divergent parts of the integrands. If instead we have used the standard dimensional regularization (i.e. have assumed that $\langle p_\alpha p_\beta \rangle = \frac{1}{d}\eta_{\alpha\beta}\langle p^2 \rangle$ instead of $\langle p_\alpha p_\beta \rangle = \frac{1}{2}\eta_{\alpha\beta}\langle p^2 \rangle$) then the contribution of the FFFF term would be $-64(1-\frac{1}{d})$ and we would be left with non-cancelled $\frac{1}{\epsilon}$ divergences (and a finite part). This indicates that the standard dimensional regularization cannot be applied to the GS action: it breaks some of its symmetries which results in non-trivial corrections to what should be a protected BPS state. This of course is not surprising given, in particular, the presence of the WZ term in the GS action.

D.2 Expansion near the S^5 solution in the light-cone gauge

The background (2.15) selects two spatial directions $x_8 \equiv \phi_2$, $x_9 \equiv \phi_3$ so a natural choice for the l.c. gauge condition that should produce a non-degenerate fermionic propagator when one expands near (2.15) is $[\Gamma_0 + \frac{1}{\sqrt{2}}(\Gamma_8 + \Gamma_9)]\theta^I = 0$. More generally, we may consider a "rotated" choice $[\Gamma_0 + \frac{1}{\sqrt{1+\zeta^2}}(\Gamma_8 + \zeta\Gamma_9)]\theta^I = 0$ where ζ is a gauge-fixing parameter. The result for the effective action does not depend on the value of ζ: since Γ_A have tangent-space indices this $=$ follows from rotational invariance of the action in the tangent space.

---

[58]Note that our computation is different from the discussions of near-BMN expansion in [39, 42] where a light-cone-type gauge was imposed on the bosons. We instead use the conformal gauge, with the conformal gauge ghosts cancelling the contribution of 2 massless longitudinal modes ($\tilde{t}$ and $\tilde{\phi}$) at 1-loop; within our regularization scheme the contribution of these modes also decouples at higher loops.

[59]We again set $\kappa = 1$ by a rescaling of 2d coordinates/momenta.

In what follows we shall choose the simplest option $\zeta = 0$, i.e.

$$\Gamma_+ \theta^I = 0 \,, \qquad\qquad\qquad \bar{\theta}^I \Gamma_+ = 0 \,, \quad (\theta^I)^T \Gamma_- = 0 \,, \qquad\qquad \text{(D.3)}$$

$$\Gamma_\pm \equiv \frac{1}{2}(\pm\Gamma_0 + \Gamma_8) \,, \quad \Gamma_-\Gamma_+ + \Gamma_+\Gamma_- = 1 \,, \qquad\quad \Gamma_\pm^2 = 0 \,, \quad (\Gamma_+\Gamma_-)^2 = \Gamma_+\Gamma_- \quad \text{(D.4)}$$

Splitting the bosonic tangent-space indices into $0, 8$ and $p, q = 1, 2, 3, 4, 5, 6, 7, 9$ we get from (A.3)[60]

$$\begin{aligned}
D_\alpha \theta^J = \partial_\alpha \theta^J &- \frac{1}{2}(\omega_\alpha^{0p} - \omega_\alpha^{8p})\Gamma_-\Gamma_p\theta^J + \frac{1}{4}\omega_\alpha^{pq}\Gamma_{pq}\theta^J \\
&+ \frac{1}{2}\epsilon^{JK} e_\alpha^p \Gamma_p \Pi \Gamma_- \theta^K - \frac{1}{2}\epsilon^{JK}(e_\alpha^0 - e_\alpha^8)\Pi\theta^K \,,
\end{aligned} \qquad \text{(D.5)}$$

where

$$\Pi \equiv \Gamma_{1234} \,, \qquad \Gamma_* = i\Gamma_0\Pi \,, \qquad \Pi^2 = 1 \,. \qquad\qquad \text{(D.6)}$$

The combination entering the quadratic fermionic term (A.8) becomes

$$\begin{aligned}
\bar{\theta}^I \slashed{e}_\alpha D_\beta \theta^J = &-(e_\alpha^0 - e_\alpha^8)\bar{\theta}^I\Gamma_-\partial_\beta\theta^J - \frac{1}{4}(e_\alpha^0 - e_\alpha^8)\omega_\beta^{pq}\bar{\theta}^I\Gamma_-\Gamma_{pq}\theta^J \\
&+ \frac{1}{2}\epsilon^{JK}(e_\alpha^0 - e_\alpha^8)(e_\beta^0 - e_\beta^8)\bar{\theta}^I\Gamma_-\Pi\theta^K \\
&- \frac{1}{2}\bar{\theta}^I e^p{}_\alpha \Gamma_p \left[(\omega_\beta^{0q} - \omega_\beta^{8q})\Gamma_-\Gamma_q\theta^J + \epsilon^{JK}\Gamma_- e_\beta^q\Gamma_q\Pi\theta^K\right] \,. \qquad \text{(D.7)}
\end{aligned}$$

Expanding the vielbein and connection near their background values in (A.11) we find for the fermionic kinetic term

$$\frac{2\pi}{\sqrt{\lambda}}\mathcal{L}_{F2}^{(0)} = i(\eta^{\alpha\beta}\delta^{IJ} - \epsilon^{\alpha\beta}\mathrm{s}^{IJ})\left[\bar{\theta}^I\Gamma_-\bar{e}_\alpha^8\partial_\beta\theta^J + \frac{1}{2}\epsilon^{JK}(\bar{e}_\alpha^8\bar{e}_\beta^8 + \bar{e}_\alpha^9\bar{e}_\beta^9)\bar{\theta}^I\Gamma_-\Pi\theta^K\right], \quad \text{(D.8)}$$

where (cf. (2.31))

$$\bar{e}_\alpha^8 = \frac{\kappa}{\sqrt{2}}(1, -i), \qquad\quad \bar{e}_\alpha^9 = \frac{\kappa}{\sqrt{2}}(1, i), \qquad \bar{e}_\alpha^8\bar{e}_\beta^8 + \bar{e}_\alpha^9\bar{e}_\beta^9 = -\kappa^2\eta_{\alpha\beta} \,, \qquad \text{(D.9)}$$

$$\eta^{\alpha\beta}\bar{e}_\alpha^8\bar{e}_\beta^8 = \eta^{\alpha\beta}\bar{e}_\alpha^9\bar{e}_\beta^9 = -1 \,, \qquad \eta^{\alpha\beta}\bar{e}_\alpha^8\bar{e}_\beta^9 = 0 \,, \qquad\qquad \epsilon^{\alpha\beta}\bar{e}_\alpha^8\bar{e}_\beta^9 = i. \qquad \text{(D.10)}$$

Thus

$$\begin{aligned}
\frac{2\pi}{\sqrt{\lambda}}\mathcal{L}_{F2}^{(0)} = \frac{\kappa}{\sqrt{2}}\Big[&(1-i)\bar{\theta}^1\Gamma_-(\partial_1 + \partial_0)\theta^1 + (1+i)\bar{\theta}^2\Gamma_-(\partial_1 - \partial_0)\theta^2 \\
&-i\sqrt{2}\kappa\left(\bar{\theta}^1\Gamma_-\Pi\theta^2 - \bar{\theta}^2\Gamma_-\Pi\theta^1\right)\Big] \equiv \frac{1}{2}\theta^T K\theta \,, \qquad \text{(D.11)}
\end{aligned}$$

where the kinetic operator in momentum representation is (we now set $\kappa = 1$)

$$K = -i\sqrt{2}\begin{pmatrix} (1-i)(q_1 + q_0) & -\sqrt{2}\Pi \\ \sqrt{2}\Pi & (1+i)(q_1 - q_0) \end{pmatrix}\Gamma_+\Gamma_- \,. \qquad \text{(D.12)}$$

---

[60]We have dropped the term with ω_α^{08} since this component of the connection vanishes for our direct-product metric.

Here we used that $\bar{\theta} = \theta^T \mathcal{C}$, $\mathcal{C} = \Gamma^0 = -\Gamma_0$ (see (A.7)) and that $\mathcal{C}\Gamma_- = (\Gamma_- - \Gamma_+)\Gamma_- = -\Gamma_+\Gamma_-$.

Then we get for the propagator (cf. (B.10))

$$K^{-1} = \frac{i}{2\sqrt{2}(q^2+1)} \begin{pmatrix} (1+i)(q_1-q_0) & \sqrt{2}\Pi \\ -\sqrt{2}\Pi & (1-i)(q_1+q_0) \end{pmatrix} \Gamma_+\Gamma_- , \quad K \cdot K^{-1} = \Gamma_+\Gamma \quad \text{(D.13)}$$

where $\Gamma_+\Gamma_- = \Gamma_+\mathcal{C}$, $q^2 = -q_0^2 + q_1^2$. The propagator can be written also in the following "covariant" form:

$$(K^{-1})^{IJ} = \frac{i}{2\sqrt{2}(q^2+1)} \left[(i\bar{e}_\alpha^8 \delta^{IJ} + \bar{e}_\alpha^9 \mathrm{s}^{IJ})q^\alpha - \sqrt{2}\Pi\epsilon^{IJ} \right] . \quad \text{(D.14)}$$

The logarithm of the determinant of K gives the same 1-loop contribution in (1.16) as found in the $\theta^1 = k\theta^2$ gauge.

The FFB and FFBB interaction vertices are found from expanding (D.7) (multiplied by $i(\eta^{\alpha\beta}\delta^{IJ} - \epsilon^{\alpha\beta}\mathrm{s}^{IJ})$ as in (A.8)) to quadratic order in bosonic fluctuation fields in (2.16) using the expressions in (A.12)–(A.20). Then the Feynman graphs are constructed using the propagators (2.24) and (D.13). For example, the interaction vertices linear in the S^5 field $\tilde{x}$ in (2.16) are given by

$$\frac{2\pi}{\sqrt{\lambda}}\mathcal{L}_{F2\ \tilde{x}} = \tilde{x}\, \mathrm{s}^{IJ}\bar{\theta}^I\Gamma_{59}\Gamma_-\theta^J + \frac{1}{\sqrt{2}}(\partial_0+i\partial_1)\tilde{x}\, \mathrm{s}^{IJ}\epsilon^{JK}\bar{\theta}^I\Gamma_{59}\Gamma_-\Pi\theta^K$$
$$- \frac{i}{2\sqrt{2}}(\partial_0+i\partial_1)\tilde{x}\, \bar{\theta}^I\Gamma_{57}\Gamma_-\theta^I - \frac{1}{2\sqrt{2}}(\partial_0-i\partial_1)\tilde{x}\, \mathrm{s}^{IJ}\bar{\theta}^I\Gamma_{57}\Gamma_-\theta^J , \quad \text{(D.15)}$$

where we used that a term with Γ_{58} similar to the one with Γ_{59} gives vanishing contribution.

The relevant 4-fermion terms follow from the general expression in (A.9). Using (A.20) ($\bar{\omega}_\alpha^{78} = \bar{e}_\alpha^8$, $\bar{\omega}_\alpha^{79} = -\bar{e}_\alpha^9$) first keeping $\bar{e}_\alpha^8, \bar{e}_\alpha^9$ general and then using relations (D.10) we find for the second term in (A.9)

$$\frac{1}{2}(\eta^{\alpha\beta}\delta^{IJ}-\epsilon^{\alpha\beta}\mathrm{s}^{IJ})(\bar{\theta}^K\Gamma^A D_\alpha\theta^K)(\bar{\theta}^I\Gamma_A D_\beta\theta^J) = \frac{1}{8}\Big[-\bar{\theta}^K\Gamma^p\Gamma_-\Gamma_7\theta^K\bar{\theta}^I\Gamma^p\Gamma_-\Gamma_7\theta^I$$
$$+i\mathrm{s}^{IJ}\epsilon^{JL}\bar{\theta}^K\Gamma^p\Gamma_-\Gamma_7\theta^K\bar{\theta}^I\Gamma_p\Gamma_9\Pi\Gamma_-\theta^L$$
$$-i\mathrm{s}^{IJ}\epsilon^{KL}\bar{\theta}^I\Gamma_p\Gamma_-\Gamma_7\theta^J\bar{\theta}^K\Gamma^p\Gamma_9\Pi\Gamma_-\theta^L$$
$$-\epsilon^{KL}\epsilon^{IM}\bar{\theta}^K\Gamma^p\Gamma_9\Pi\Gamma_-\theta^L\bar{\theta}^I\Gamma_p\Gamma_9\Pi\Gamma_-\theta^M \Big] .$$
$$\text{(D.16)}$$

The first term in (A.9) contains two structures:

$$\bar{\theta}^I\slashed{\ell}_\alpha\mathcal{M}_{JK}^2 D_\beta\theta^K = \bar{e}_\alpha^8\bar{\theta}^I\Gamma_-\ \mathcal{M}_{JK}^2 D_\beta\theta^K + \bar{e}_\alpha^9\bar{\theta}^I\Gamma_9\mathcal{M}_{JK}^2 D_\beta\theta^K . \quad \text{(D.17)}$$

Computing them using (A.4), (D.10) we get

$$(\eta^{\alpha\beta}\delta^{IJ} - \epsilon^{\alpha\beta}\mathrm{s}^{IJ})\bar{e}_\alpha^8\bar{\theta}^I\Gamma_-\,\mathcal{M}^2_{JK}D_\beta\theta^K$$

$$= \frac{i}{2}(\eta^{\alpha\beta}\delta^{IJ} - \epsilon^{\alpha\beta}\mathrm{s}^{IJ})\bar{e}_\alpha^8\epsilon^{LK}\big[\bar{\theta}^I\Gamma_-\Gamma^{ij}\theta^J\bar{\theta}^L\Gamma_{ij}\Gamma_-\Pi\partial_\beta\theta^K - \bar{\theta}^I\Gamma_-\Gamma^{i'j'}\theta^J\bar{\theta}^L\Gamma_{i'j'}\Gamma_-\Pi\partial_\beta\theta^K\big]$$

$$\quad -\frac{1}{4}\mathrm{s}^{IJ}\epsilon^{LK}\big[\bar{\theta}^I\Gamma_-\Gamma^{ij}\theta^J\bar{\theta}^L\Gamma_{ij}\Gamma_-\Pi\Gamma_{79}\theta^K - \bar{\theta}^I\Gamma_-\Gamma^{i'j'}\theta^I\bar{\theta}^L\Gamma_{i'j'}\Gamma_-\Pi\Gamma_{79}\theta^K\big]$$

$$\quad +\frac{i}{4}\big[\bar{\theta}^I\Gamma_-\Gamma^{ij}\theta^I\bar{\theta}^L\Gamma_{ij}\Gamma_-\theta^L - \bar{\theta}^I\Gamma_-\Gamma^{i'j'}\theta^I\bar{\theta}^L\Gamma_{i'j'}\Gamma_-\theta^L\big]\,, \tag{D.18}$$

$$(\eta^{\alpha\beta}\delta^{IJ} - \epsilon^{\alpha\beta}\mathrm{s}^{IJ})\bar{e}_\alpha^9\bar{\theta}^I\Gamma_9\mathcal{M}^2_{JK}D_\beta\theta^K$$

$$= \frac{1}{2}\mathrm{s}^{IJ}\big[\epsilon^{JL}\bar{\theta}^I\Gamma_9\Gamma_-\Pi\Gamma^p\theta^L\bar{\theta}^K\Gamma_p\Gamma_-\Gamma_7\theta^K$$

$$\quad +\bar{\theta}^I\Gamma_9\epsilon^{LK}\Gamma_-\Gamma^i\theta^J\bar{\theta}^L\Gamma_i\Pi\Gamma_-\Gamma_7\theta^K - \epsilon^{LK}\bar{\theta}^I\Gamma_9\Gamma_-\Gamma^{i'}\theta^J\bar{\theta}^L\Gamma_{i'}\Pi\Gamma_-\Gamma_7\theta^K\big]$$

$$\quad -\frac{i}{2}\big[\epsilon^{IL}\epsilon^{KM}\bar{\theta}^I\Gamma_9\Gamma_-\Pi\Gamma^p\theta^L\bar{\theta}^K\Gamma_p\Gamma_9\Pi\Gamma_-\,\theta^M$$

$$\quad +\bar{\theta}^I\Gamma_9\Gamma_-\Gamma^i\theta^I\bar{\theta}^L\Gamma_i\Gamma_9\Gamma_-\theta^L - \bar{\theta}^I\Gamma_9\Gamma_-\Gamma^{i'}\theta^I\bar{\theta}^L\Gamma_{i'}\Gamma_9\Gamma_-\theta^L\big]\,, \tag{D.19}$$

where $i,j = 1,2,3,4$; $i',j' = 5,6,7,9$ and $p = (i,i')$.

E. Calculation of 2-loop momentum integrals

E.1 Bosonic integrals

Here we compute the integral of $\mathcal{I}_N$ in (2.30) that enters (2.26) and (2.36). We split the integrals in the same way as their integrands in (2.30)

$$I_N = 3(I_{N,1} + I_{N,2} + I_{N,3})\,, \qquad I_{N,i} = \int \frac{d^2q_i d^2q_j}{(2\pi)^4}\,\mathcal{I}_{N,i}\,. \tag{E.1}$$

Let us start with $\mathcal{I}_{N,1}$ and introduce the tensor

$$I_1^{\alpha\beta\gamma\delta} = \int \frac{d^2q_i d^2q_j d^2q_k}{(2\pi)^4}\,\delta^{(2)}(q_i + q_j + q_k)\,\frac{q_i^\alpha q_j^\beta q_i^\gamma q_j^\delta\,(q_i^2 + q_j^2 - q_k^2)^2}{(q_i^2)^2\,(q_i^2 + 4)\,(q_j^2)^2\left(q_j^2 + 4\right)(q_k^2 + 4)}$$

$$= \frac{1}{4}\big[A_1\,\eta^{\alpha\gamma}\eta^{\beta\delta} + A_2\,(\eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma})\big]\,, \tag{E.2}$$

where we used the symmetry under $q_i \leftrightarrow q_j$.[61] Taking traces over (α,γ) and (β,δ) we obtain

$$A_1 + A_2 = \int \frac{d^2q_i d^2q_j d^2q_k}{(2\pi)^4}\,\delta^{(2)}(q_i + q_j + q_k)\,\frac{(q_i^2 + q_j^2 - q_k^2)^2}{q_i^2\left(q_i^2 + 4\right)q_j^2\left(q_j^2 + 4\right)(q_k^2 + 4)}\,. \tag{E.3}$$

---

[61] We reinstated the integral over q_k to make the symmetry between q_i and q_j manifest. Also, we used the notation $\eta_{\alpha\beta}$ for the 2d metric. The integrand (2.30) was already continued to Euclidean space; at the level of the above analysis this replaces $\eta_{\alpha\beta}$ with $\delta_{\alpha\beta}$.

We only need that particular combination of A_1 and A_2 to compute $I_{N,1}$. Expanding the numerator and using various symmetric integration identities we get from (E.3)

$$
\begin{aligned}
I_{N,1} = 4(A_1 + A_2) = 4 \int \frac{d^2 q_i d^2 q_j}{(2\pi)^{2d}} \Bigg[&- \frac{4}{q_i^2(q_i^2+4)q_j^2(q_j^2+4)} \\
&+ \frac{2}{q_i^2(q_i^2+4)(q_k^2+4)} + \frac{2}{q_i^2(q_j^2+4)(q_k^2+4)} \\
&- \frac{2}{q_i^2(q_i^2+4)(q_j^2+4)} + \frac{16}{q_i^2(q_i^2+4)q_j^2(q_j^2+4)(q_k^2+4)} \Bigg] \\
= 4 \int \frac{d^2 q_i d^2 q_j}{(2\pi)^4} \Bigg[&- \frac{4}{q_i^2(q_i^2+4)q_j^2(q_j^2+4)} \\
&+ \frac{1}{q_i^2 q_j^2[(q_i+q_j)^2+4]} + \frac{1}{(q_i^2+4)(q_j^2+4)[(q_i+q_j)^2+4]} \Bigg]. \text{(E.4)}
\end{aligned}
$$

For $I_{N,2}$ we proceed in the same way by starting with the tensor

$$
\begin{aligned}
I_2^{\alpha\beta\gamma\delta} &= \int \frac{d^2 q_i d^2 q_j d^2 q_k}{(2\pi)^4} \, \delta^{(2)}(q_i + q_j + q_k) \, \frac{q_i^\alpha q_i^\beta q_i^\gamma q_i^\delta}{q_i^4(q_i^2+4)(q_j^2+2)(q_k^2+2)} \\
&= \frac{1}{8} A_3 \left(\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} \right),
\end{aligned} \tag{E.5}
$$

where A_3 is found by taking the trace. As a result,

$$
I_{N,2} = -8 A_3 = -8 \int \frac{d^2 q_i d^2 q_j}{(2\pi)^4} \frac{1}{(q_i^2+4)(q_j^2+2)[(q_i+q_j)^2+2]} . \tag{E.6}
$$

For the integral in the last term $\mathcal{I}_{N,3}$ in (2.30) we need to consider two tensors associated with the prefactor

$$
-(q_{i0}q_{j0} - q_{i1}q_{j1})(q_{i0}q_{k0} - q_{i1}q_{k1}) = (q_{i0}q_{j0} - q_{i1}q_{j1})^2 + (q_{i0}q_{j0} - q_{i1}q_{j1})(q_{i0}^2 - q_{i1}^2), \tag{E.7}
$$

i.e. one with two q_i's and two q_j's and the other one with three q_i's and one q_j. The first one is then similar to $I_1^{\alpha\beta\gamma\delta}$ in (E.2)

$$
\begin{aligned}
I_3^{\alpha\beta\gamma\delta} &= \int \frac{d^2 q_i d^2 q_j d^2 q_k}{(2\pi)^4} \, \delta^{(2)}(q_i + q_j + q_k) \, \frac{q_i^\alpha q_j^\beta q_i^\gamma q_j^\delta \, [(q_i^2)^2 - (q_j^2 - q_k^2)^2]}{(q_i^2)^2(q_i^2+4)q_j^2(q_j^2+4)q_k^2(q_k^2+4)} \\
&= \frac{1}{4} \left[A_4 \, \eta^{\alpha\gamma}\eta^{\beta\delta} + A_5 \left(\eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} \right) \right],
\end{aligned} \tag{E.8}
$$

where

$$
\begin{aligned}
A_4 + A_5 &= \int \frac{d^2 q_i d^2 q_j d^2 q_k}{(2\pi)^4} \, \delta^{(2)}(q_i + q_j + q_k) \, \frac{[(q_i^2)^2 - (q_j^2 - q_k^2)^2]}{q_i^2(q_i^2+4)(q_j^2+4)q_k^2(q_k^2+4)} \\
&= \int \frac{d^2 q_i d^2 q_j}{(2\pi)^4} \Bigg[\frac{4}{q_i^2(q_i^2+4)q_j^2(q_j^2+4)} \\
&\quad - \frac{1}{q_i^2 q_j^2[(q_i+q_j)^2+4]} + \frac{1}{(q_i^2+4)(q_j^2+4)[(q_i+q_j)^2+4]} \Bigg]. \tag{E.9}
\end{aligned}
$$

The second tensor we need is

$$
\begin{aligned}
\tilde{I}_3^{\alpha\beta\gamma\delta} &= \int \frac{d^2q_i d^2q_j d^2q_k}{(2\pi)^4}\, \delta^{(2)}(q_i+q_j+q_k)\, \frac{q_i^\alpha q_i^\beta q_i^\gamma q_i^\delta\,[(q_i^2)^2-(q_j^2-q_k^2)^2]}{(q_i^2)^2(q_i^2+4)q_j^2(q_j^2+4)q_k^2(q_k^2+4)} \\
&= \frac{1}{8}A_6\,(\eta^{\alpha\gamma}\eta^{\beta\delta}+\eta^{\alpha\beta}\eta^{\gamma\delta}+\eta^{\beta\gamma}\eta^{\alpha\delta}) + A_7\,(\eta^{\alpha\gamma}\epsilon^{\beta\delta}+\eta^{\alpha\beta}\epsilon^{\gamma\delta}+\eta^{\beta\gamma}\epsilon^{\alpha\delta})\,. \quad (E.10)
\end{aligned}
$$

The A_7 term is not contributing in our case since the combination in (E.7) is symmetric in q_i, q_j (in fact, $A_7 = 0$ as one can see by doing explicitly one of the two integrals). Taking traces gives

$$
\begin{aligned}
A_6 &= \frac{1}{2}\int \frac{d^2q_i d^2q_j d^2q_k}{(2\pi)^4}\,\delta^{(2)}(q_i+q_j+q_k)\,\frac{(q_k^2-q_i^2+q_j^2)\,[(q_i^2)^2-(q_j^2-q_k^2)^2]}{q_i^2(q_i^2+4)q_j^2(q_j^2+4)q_k^2(q_k^2+4)} \\
&= -\frac{1}{2}\int \frac{d^2q_i d^2q_j}{(2\pi)^4}\Bigg[-\frac{4}{q_i^2(q_i^2+4)q_j^2(q_j^2+4)} \\
&\quad + \frac{1}{q_i^2 q_j^2[(q_i+q_j)^2+4]} + \frac{1}{(q_i^2+4)(q_j^2+4)[(q_i+q_j)^2+4]} \Bigg] \quad (E.11)
\end{aligned}
$$

Finally, we get

$$
I_{N,3} = 4\,(A_4+A_5+2A_6)\,. \quad (E.12)
$$

Summing up the above expressions (E.4), (E.6) and (E.12) we obtain for I_N in (E.1)

$$
\begin{aligned}
I_N = 24 \int \frac{d^2q_i d^2q_j}{(2\pi)^4}\Bigg[&\frac{4}{q_i^2(q_i^2+4)q_j^2(q_j^2+4)} - \frac{1}{q_i^2 q_j^2[(q_i+q_j)^2+4]} \\
&- \frac{1}{(q_i^2+4)(q_j^2+2)[(q_i+q_j)^2+2]} + \frac{1}{(q_i^2+4)(q_j^2+4)[(q_i+q_j)^2+4]} \Bigg] \quad (E.13)
\end{aligned}
$$

The integrands on the first line of (E.13) combine into $\frac{8q_i\cdot q_j}{q_i^2 q_j^2(q_i^2+4)(q_j^2+4)[(q_i+q_j)^2+4]}$ and the resulting IR finite integral can be evaluated using Feynman parametrization. Alternatively, we may evaluate the two integrals separately introducing an IR cutoff $m_0 \to 0$ and using that

$$
\int \frac{d^2q_i d^2q_j}{(2\pi)^4}\left(\frac{1}{q_i^2+m_0^2}-\frac{1}{q_i^2+4}\right)\left(\frac{1}{q_j^2+m_0^2}-\frac{1}{q_j^2+4}\right) \;\to\; \frac{1}{(4\pi)^2}\ln^2\left(\frac{m_0^2}{4}\right), \quad (E.14)
$$

and also the previously computed expression (2.45) for (2.41) (see (2.42), (2.44)), i.e.

$$
\begin{aligned}
\int \frac{d^2q_i d^2q_j}{(2\pi)^4}\frac{1}{(q_i^2+m_0^2)(q_j^2+m_0^2)[(q_i+q_j)^2+4]} &= \frac{1}{(4\pi)^2}\int_0^1 dx\frac{\ln\frac{4}{m_0^2}+\ln[x(1-x)]}{4x(1-x)-m_0^2} \\
&\to \frac{1}{4(4\pi)^2}\left[\frac{13}{3}\pi^2+\ln^2\left(\frac{m_0^2}{4}\right)\right]. \quad (E.15)
\end{aligned}
$$

The remaining two integrals in (E.13) are again of the familiar type (2.42), (2.44) and are the same as in (2.46) and (2.49)

$$
\int \frac{d^2q_i d^2q_j}{(2\pi)^4}\frac{1}{(q_i^2+4)(q_j^2+4)[(q_i+q_j)^2+4]} = \frac{1}{4(4\pi)^2}\int_0^1 dx\frac{\ln[x(1-x)]}{x(1-x)-1}, \quad (E.16)
$$

$$
\int \frac{d^2q_i d^2q_j}{(2\pi)^4}\frac{1}{(q_i^2+4)(q_j^2+2)[(q_i+q_j)^2+2]} = \frac{1}{2(4\pi)^2}\int_0^1 dx\frac{\ln[2x(1-x)]}{2x(1-x)-1}\,. \quad (E.17)
$$

They are thus expressed in terms of the Catalan constant K (2.47) and a combination of trigamma values $\tilde{K}$ (2.50). Explicitly, combining the values of the above integrals we find for (E.13)

$$I_N = -\frac{13}{8} - \frac{24}{(4\pi)^2}(K - \tilde{K}) .\qquad(E.18)$$

E.2 Fermionic integrals

The non-invariant integral in the mixed boson-fermion sector contains two different types of factors. The first is (here we use Euclidean signature and consider the integral directly in $d = 2$):

$$X = (q_{i0}^2 - q_{i1}^2)^2 = (q_i^2 - 2q_{i1}^2)^2 \qquad(E.19)$$

and its expectation value over (q_i, q_j) symmetric Lorentz-invariant measure can be evaluated using that as in (E.5) $\langle q_i^\alpha q_i^\beta q_i^\gamma q_i^\delta \rangle = \frac{1}{8}(\eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\gamma}\eta^{\delta\beta} + \eta^{\alpha\delta}\eta^{\gamma\beta})\langle q_i^4 \rangle$. This gives

$$\langle X_1 \rangle = \frac{1}{2}\langle q_i^4 \rangle .\qquad(E.20)$$

The second combination is

$$Y = (q_{i0}q_{j0} - q_{i1}q_{j1})(q_{k0}q_{k0} - q_{k1}q_{k1}) = (q_i^2 - 2q_{i1}q_{j1})(q_k^2 - 2q_{k1}q_{k1}) , \quad q_k = -q_i - q_j .\qquad(E.21)$$

Using the $q_i \to q_j$ symmetry of the measure the expectation value of X_2 is the same as of

$$Y' = 2q_i \cdot q_j(q_i^2 + q_i \cdot q_j) - 4q_{i1}q_{j1}(q_i^2 + q_i \cdot q_j) - 4(q_{i1}^2 + q_{i1}q_{j1}) + 8q_{i1}q_{j1}(q_{i1}^2 + q_{i1}q_{j1}) .\qquad(E.22)$$

Then $\langle Y' \rangle$ can be found by using the same relations as in (E.10), (E.8)

$$\langle q_i^\alpha q_i^\beta q_i^\gamma q_j^\delta \rangle = \frac{1}{8}\langle q_i^2(q_i \cdot q_j)\rangle(\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\beta\gamma}\eta^{\alpha\delta}) ,\qquad(E.23)$$

$$\begin{aligned}\langle q_i^\alpha q_j^\beta q_i^\gamma q_j^\delta \rangle = \frac{1}{8}\big[&\langle -2(q_i \cdot q_j)^2 + 3q_i^2 q_j^2\rangle\eta^{\alpha\gamma}\eta^{\beta\delta} \\ &+ \langle 2(q_i \cdot q_j)^2 - q_i^2 q_j^2\rangle\ (\eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma})\big] .\end{aligned}\qquad(E.24)$$

As a result,

$$\langle Y \rangle = \langle Y' \rangle = \langle (q_i \cdot q_j)^2 + q_i^2 q_j^2 \rangle = \frac{1}{4}\langle (q_i^2 + q_j^2)(q_i + q_j)^2 - (q_i^2 - q_j^2)^2 \rangle .\qquad(E.25)$$

Let us now consider again the similar integrals in d dimensions keeping track of d-dependent factors.[62] Here we shall use Minkowski signature and always imply that $q_i + q_j + q_k = 0$. We start with

$$\int d^d q_j d^d q_k\, q_k^a q_k^b q_k^c q_k^d\, f(q_j, q_k) = A\,(\eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma})\qquad(E.26)$$

$$A = \frac{1}{d(d+2)}\int d^d q_j d^d q_k\, (q_k^2)^2\, f(q_j, q_k) .\qquad(E.27)$$

---

[62]That may be useful for finding the coefficient of the $\frac{1}{\epsilon}$ divergences in the fermionic sector as in the bosonic sector in (2.36), (2.37).

In particular, we find

$$\int d^d q_j d^d q_k \, (q_{k0}^2 + q_{k1}^2)^2 \, f(q_j, q_k) = 4A \; . \tag{E.28}$$

Let us consider the following combination

$$2(q_{i0}q_{j0} + q_{i1}q_{j1})(q_{k0}^2 + q_{k1}^2) = -(q_{i0}^2 + q_{i1}^2)(q_{k0}^2 + q_{k1}^2) - (q_{j0}^2 + q_{j1}^2)(q_{k0}^2 + q_{k1}^2) \tag{E.29}$$
$$-(q_{i0}q_{k0} + q_{i1}q_{k1})(q_{k0}^2 + q_{k1}^2) - (q_{j0}q_{k0} + q_{j1}q_{k1})(q_{k0}^2 + q_{k1}^2)$$

The reason for this splitting is to maintain the $i \leftrightarrow j$ symmetry. To evaluate its integral we will need

$$\int d^d q_j d^d q_k \, q_i^a q_i^b q_k^c q_k^d \, f(q_j, q_k) = A_i \eta^{ab} \eta^{cd} + B_i (\eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc}) \tag{E.30}$$

Then

$$d^2 \, A_i + 2d \, B_i = \int d^d q_j d^d q_k \, q_i^2 q_k^2 \, f(q_i, q_k),$$
$$d \, A_i + d(d+1) \, B_i = \int d^d q_j d^d q_k \, (q_i \cdot q_k)^2 \, f(q_i, q_k) \,, \tag{E.31}$$

and thus

$$4B_i = \int d^d q_j d^d q_k \, (q_{i0}^2 + q_{i1}^2)(q_{k0}^2 + q_{k1}^2) \, f(q_j, q_k)$$
$$= \frac{4}{d(d+1) - 2} \int d^d q_j d^d q_k \left[(q_i \cdot q_k)^2 - \frac{1}{d} \, q_i^2 \, q_k^2 \right] f(q_i, q_k) \tag{E.32}$$

Consider also

$$\int d^d q_j d^d q_k \, q_j^a q_j^b q_k^c q_k^d \, f(q_j, q_k) = D_j (\eta^{ab} \eta^{cd} + \eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc})$$
$$\int d^d q_{i,j} d^d q_k \, (q_{i,j0}q_{k0} + q_{i,j1}q_{k1})(q_{k0}^2 + q_{k1}^2) \, f(q_{i,j}, q_k)$$
$$= 4D_{i,j} = \frac{4}{d(d+2)} \int d^d q_{i,j} d^d q_k (q_{i,j} \cdot q_k) \, q_k^2 \, f(q_{i,j}, q_k) \,. \tag{E.33}$$

Collecting separate terms we get

$$2 \int d^d q_i d^d q_j d^d q_k \delta^d(q_i + q_j + q_k)(q_{i0}q_{j0} + q_{i1}q_{j1})(q_{k0}^2 + q_{k1}^2) f(q_i, q_j, q_k) \tag{E.34}$$
$$= \int d^d q_i d^d q_j d^d q_k \delta^d(q_i + q_j + q_k) \left\{ \frac{4}{d(d+1) - 2} \left[(q_i \cdot q_k)^2 + (q_j \cdot q_k)^2 - \frac{1}{d}(q_i^2 + q_j^2) q_k^2 \right] \right.$$
$$\left. + \frac{4}{d(d+2)} (q_i \cdot q_k + q_j \cdot q_k) q_k^2 \right\} f(q_i, q_j, q_k)$$

Using the momentum conservation $q_i + q_j + q_k = 0$ we can reorganize various terms:

$$(q_i \cdot q_k)^2 + (q_j \cdot q_k)^2 + \frac{1}{2}(q_i \cdot q_k + q_j \cdot q_k) q_k^2 = \frac{1}{2}(q_k^2 + 2q_i \cdot q_k) q_i \cdot q_k + \frac{1}{2}(q_k^2 + 2q_j \cdot q_k) q_j \cdot q_k$$
$$= \frac{1}{2}(q_j^2 - q_i^2) q_i \cdot q_k + \frac{1}{2}(q_i^2 - q_j^2) q_j \cdot q_k$$
$$= \frac{1}{2}(q_i^2 - q_j^2)^2 \tag{E.35}$$

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and then

$$2 \int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k)(q_{i0}q_{j0} + q_{i1}q_{j1})(q_{k0}^2 + q_{k1}^2)f(q_i, q_j, q_k)$$
$$= \int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k)$$
$$\left\{ \frac{2}{d(d+1)-2}\left[-(q_i^2 - q_j^2)^2 + \frac{2}{d}(q_i^2 + q_j^2)q_k^2 \right] - \frac{2(2-d)}{d(d(d+1)-2)}(q_k^2)^2 \right\} f(q_i, q_j, q_k)$$

Similarly, we can compute the integrals

$$I_1 = \int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k)(q_{i0}q_{j1} + q_{i1}q_{j0})(q_{i0}q_{k1} + q_{i1}q_{k0})f(q_i, q_j) \quad \text{(E.36)}$$

$$I_2 = \int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k)(q_{i0}q_{j1} + q_{i1}q_{j0})(q_{i0}q_{k1} + q_{i1}q_{k0})f(q_i, q_k) \quad \text{(E.37)}$$

$$I_3 = \int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k)(q_{i0}q_{j1} + q_{i1}q_{j0})(q_{i0}q_{k1} + q_{i1}q_{k0})f(q_i, q_j, q_k) \quad \text{(E.38)}$$

Let us consider

$$\int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k) q_i^a q_j^b q_i^c q_j^d f(q_i, q_j) = \frac{1}{4}[A_1 \eta^{ac}\eta^{bd} + A_2(\eta^{ab}\eta^{cd} + \eta^{ad}\eta^{bc})] \quad \text{(E.39)}$$

We obtain

$$4 \int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k) q_i^2 q_j^2 f(q_i, q_j) = A_1 d^2 + 2A_2 d \quad \text{(E.40)}$$

$$\int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k)(q_i^2 + q_j^2 - q_k^2)^2 f(q_i, q_j) = A_1 d + A_2 d(d+1) \quad \text{(E.41)}$$

Then A_1 and A_2 are

$$A_1 = \frac{1}{d^2(d+1)-2d} \int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k)[4(d+1)q_i^2 q_j^2 - 2(q_i^2 + q_j^2 - q_k^2)^2]f(q_i, q_j)$$
$$\text{(E.42)}$$

$$A_2 = \frac{1}{d^2(d+1)-2d} \int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k)[-4q_i^2 q_j^2 + d(q_i^2 + q_j^2 - q_k^2)^2]f(q_i, q_j)$$
$$\text{(E.43)}$$

Also

$$\int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k) q_i^\alpha q_i^\beta q_i^\gamma q_j^\delta f(q_i, q_j) = \frac{1}{8}A_3[\eta^{ac}\eta^{\beta\delta} + \eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma}] \quad \text{(E.44)}$$

from which one obtains

$$8 \int d^d q_i d^d q_j d^d q_k \delta^{(d)}(q_i + q_j + q_k) q_i^2 q_i q_j f(q_i, q_j) = d(d+2)A_3 \quad \text{(E.45)}$$

The integral I_1 becomes

$$I_1 = -\frac{1}{2}(A_1 + A_2 + A_3) \quad \text{(E.46)}$$

The integral I_2 can be written in the same way as I_1 with the formal interchanging $j \leftrightarrow k$ in A_1, A_2, A_3. The integral I_3 is the same as I_1.

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